Statistics –I

THE CENTRAL LIMIT THEOREM

In probablity theory there is a very elegent theorem called the central limit theorem.A special case of this theorem asserts that if $x1....x_n$ denote the items of a random sample of size from any distribution Having positive variance σ^2,then the random Variable $\sqrt{n}\,(\bar{x} - \mu)/\sigma$ has a limiting normal Distribution with mean 0 and variance 1.

the more general form of the theorem is stated but it is proved only in the modifed case.

THEOREM:3

Let $x_1, x_2 \ldots . x_n$ denote the items of a random sample from a distribution that has mean μ and positive variance σ^2 then the random variable.

$$y_n = (\Sigma^n{}_1\, x_i - n\mu)/\sigma\sqrt{n} = \sqrt{n}(\bar{x} - \mu)/\sigma$$

has a limiting normal distribution with mean 0 and variance 1.

PROOF:-

We assume the existance of the m.g.f $M(t)=E(e^{tx})$, $-h<t<h$ of the distribution.

However this proof is essentially the same one that would be given if we could use the characteristic function in place of the m.g.f

The function $M(t)=E\ [e^{t(X-\mu)}]=e^{-\mu t}\ E\ [e^{tx}\ ]$ $=e^{-\mu t}$ also exist for $-h<t<h$.

since $M(t)$ is the m.g.f for $(x-\mu)$

We have $m(0)=1$,

$m'(0)=E(X-\mu)$ and $m''(0)=E[(X-\mu)^2]=\sigma^2$

By taylor's formula,

there exists a number ξ between 0 and t such that

$$m(t)=m(0)+m'(0)+(m''(\xi)t^2)/2$$

$$=1+ (m''(\xi)t^2)/2$$

If $\sigma^2 t^2/2$ is added and subtracted, then

$$m(t)=1+ \sigma^2 t^2/2 +[(m''(\xi)- \sigma^2)\, t^2]/2.$$

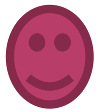
Next consider M(t;n) , where

$$M(t;n)=E\left[\exp\left(t\,(\sum X_i-n\mu)/\sigma\sqrt{n}\right) \right]$$

$$=E\left[\exp\left(t\,(X_1-\mu)/\sigma\sqrt{n}\right) .\ \exp\left(t\,(X_2-\mu)/\sigma\sqrt{n}\right) \ldots\ldots \exp\left(t\,(X_n-\mu)/\sigma\sqrt{n}\right) \right]$$

$$=E\left[\exp\left(t\,(X_1-\mu)/\sigma\sqrt{n}\right) \right] \ldots\ldots\ldots\ldots E\left[\exp\left(t\,(X_n-\mu)/\sigma\sqrt{n}\right) \right]$$

$$=\left\{ E\left[\exp\left(t\,(X-\mu)/\sigma\sqrt{n}\right) \right] \right\}^2=\left[m\,(t/\sigma\sqrt{n}) \right]^2$$

In equation  replace t by $t/\sigma\sqrt{n}$ to obtain

$$m\,(t/\sigma\sqrt{n}) = 1 + t^2/2n + [(m''(\xi) - \sigma^2)\, t^2]/2n\sigma^2$$

Where now ξ is between 0 and $t/\sigma\sqrt{n}$ with

$-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$. Accordingly ,

$$M(t;n) = \{ 1 + t^2/2n + [(m''(\xi) - \sigma^2)\, t^2]/2n\,\sigma^2 \}^n$$

Since $m''(t)$ is continuous at $t=0$ and since $n \to \infty$,

We have

$$\lim_{n \to \infty} [m''(t) - \sigma^2] = 0.$$

The limit proposition shows that,

$$\lim_{n \to \infty} M(t;n) = e^{t^2/2}$$

For all real values of t. this proves that the

random variable $Y_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution.

We interpret this theorem as saying that, when n is a large integer, the random variable $\bar{X}$ has an approximate normal distribution with mean μ and variance σ^2/n; and in applications

we use the approximate normal p.d.f As through it were the exact p.d.f of X

EXAMPLES:-

Example:1

let X denote the mean of a random Sample of size 75 from the distribution has the p.d.f

$f(x) = 1, \quad 0 < x < 1$

$\quad = 0 \quad$ elsewhere.

Example:2

let $X_1 . X_2 \ldots\ldots X_n$ denote a random Sample from a distribution that is b(1,p).

Here $\mu=p$, $\sigma^2=p(1-p)$,and M(t) exists for all real Values of t. if $Y_n= X_1 +X_2 +\ldots.+X_n,$

Example:3

with the background of example 2, let n=100 and p=1, and suppose that we wish to compute p_r (Y=48,49,50,51,52)

Example:4

let Y_n (Y for simplicity) be b(n,p). Thus Y/n is approximately N(p.p(1-p)/n)]. Statisticians often look for functions of statistics whose variances do not depnd upon the parameter.Here the variance of Y/n depends upon p.Can we find a function, say u(Y,n), whose variance is essentially free of p?

THANK YOU