AN INTRODUCTION TO

BUSINESS MATHEMATICS

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P19

(ii) BBA.

ALLIED COURSE – 3
PAPER – 5BUSINESS MATHEMATICS

Objectives:

At the end of this course the student will be able to

1. *Explain basic methods of Analytical Geometry, Set theory, business calculus, and their basic applications in practice,*
2. *Discern effects of various types and methods of simple and compound interest account.*
3. *Connect the acquired knowledge and -skills with practical problems in economics*

Unit I - ANALYTICAL GEOMETRY

Analytical Geometry – Distance between two points in a plane – Slope of a straight line – Equation of a straight line – Point of intersection of two lines – Cost P/O curves – Demand & Supply curves – Break even analysis.

Unit II - SETS

Sets – Basic concepts- Types – Subsets-Operation on sets –Venn diagram – Laws of sets - applications .

Unit III - MATRICES

Matrices – Basic Concepts – Addition of Matrices – Scalar and Matrix Multiplication-Inverse – Solution of a system of Linear equations-Matrix inversion technique, Crammer's rule.

Unit IV – DIFFERENTIAL CALCULAS

Differential calculus – Limit – Continuity – Related Variables – Average and Marginal Concept – Differential Co-efficient – Standard Forms – Differentiation: Concept and rules – Higher order derivatives – Increasing and decreasing functions – Criteria for Maxima and Minima – Applications.

(P.T.O)

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UNIT – 5 - **PERCENTAGES**

Percentages – Discount – Trade Discount – *Cash Discount – Simp*[illegible]*nd compound Inte*[illegible] Interest – True and Bankers Discount.

20% of the Questions must be theory.
80% of the Questions must be problems.

Text Book:

1. V.Sundaresan, S.D.Jeyaseelan – An Introduction to Business Mathematics – Reprin[illegible] – 2015– S.Chand and Co., Ltd. ISBN 81-219-1463-9.

[illegible]ference Books:

1. D.C.Sancheti, V.K.Kapoor – Business Mathematics – 11th edition Reprint 2014 – Sultan Chand and sons. ISBN-978-81-8054-538-2.
2. JK. Sharma –Business Mathematics Theory And Applications- 2009 - ANE Books 13th Edition –ISBN-978-8180521836

Books :-

1) V. Sundareesan, S.D. Jeyaseelan – An introduction to Business mathematics.

2) Dr. M. Manoharan, Dr. C. Elango, prof. K.L. Eswaran } Business mathematics.

I

Analytical Geometry

1·1. Introduction

There are many problems in geometry which can be translated into algebra, solved using algebraic techniques, and translated back into geometry. Frequently such problems are difficult to solve by geometrical reasoning alone. There are algebraic problems, of course, for which drawing a picture may very much help us to have an insight into the problems. The French mathematician Ren'e Descartes (1596—1650) gave an algebraic method of discussing the geometrical properties of curves.

We consider any plane and take the most commonly used system—the rectangular co-ordinate system. Any two straight lines in

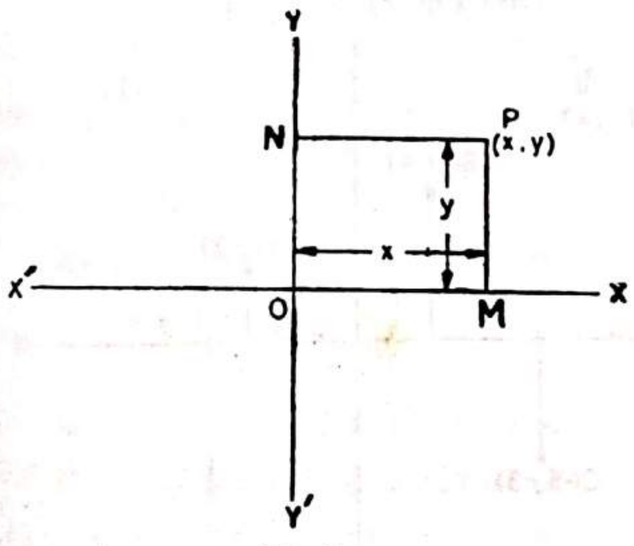


Fig. 1.

the plane intersecting at right angles are used as lines of reference and a point is located in the plane by giving its distance from each of them. The lines of reference are called the **Coordinate Axes** or briefly the axes and the point of intersection, the **Origin**. Just for convenience we take the lines to be horizontal and vertical and call them *X*-axis and *Y*-axis respectively (ref. Fig. 1). The plane is divided into four parts known as quadrants. The *x*-**Co-ordinate** (also called abscissa) of a point is its distance from its *y*-axis and it can be taken positive if it is on the right and negative if on the left of the *y*-axis. The *y*— **Co-ordinate** (also called ordinate) of the point is its distance from the *x*-axis and it is considered positive if it is above and negative if below the *x*-axis. Thus *NP* gives the *x*-co-ordinate and *MP* the *y*-co-ordinate of the point *P* (Fig. 1). Now, *OM* the distance moved along the *x*-axis is equal to *NP* and *MP* is nothing but the distance moved parallel to the *Y*-axis to reach *P*. Thus *OM* and *MP* can be taken as the *x* and *y* co-ordinates. We denote the point *P* by $P(x, y)$, $x = OM$ giving the *x*-co-ordinate and $y = MP$ giving the *y*-co-ordinate. The order in which we write *x* and *y* is very important. That is why (x, y) is called an ordered pair. The first number gives the distance we have to travel along the *x*-axis and the second number the distance parallel to the *Y*-axis. Thus, the point (2, 3) is different from the point (3, 2). The points having both the co-ordinates as positive numbers lie in the I quadrant. When both the co-ordinates of a point are negative the point is in the III quadrant. If the *x*-co-ordinate of a point is negative and the *y*-co-ordinate of the point is positive then the point is in the II quadrant. For a point in the IV quadrant the *x*-co-ordinate is positive and *y*-co-ordinate is negative (Refer Fig. 2).

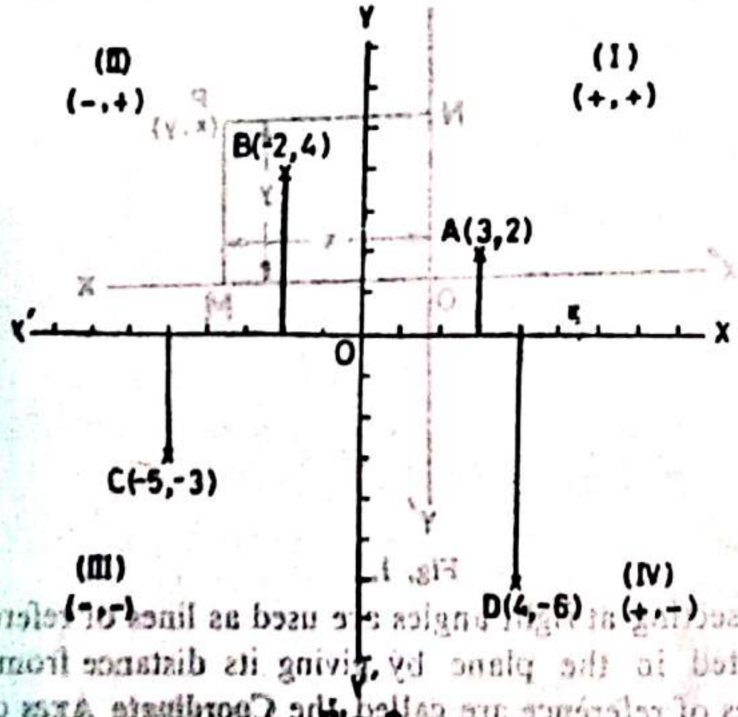


Fig. 2

1·2. Distance between Two Points in a Plane

We establish a formula for finding the distance between two points.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points. For convenience we take the points in the first quadrant and establish the formula. The formula will be true wherever A and B may be. We draw perpendiculars AM, BN to the x-axis and AR perpendicular to BN.

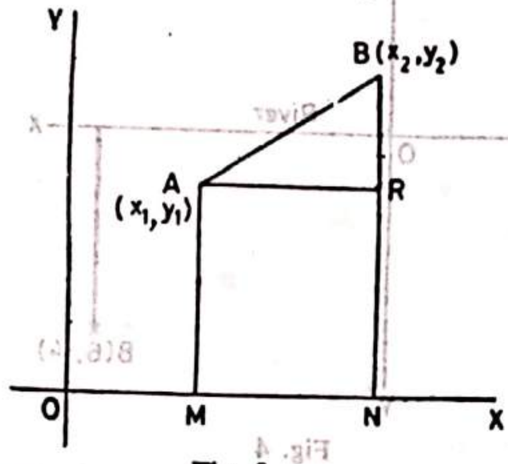


Fig. 3

$$AR = MN = ON - OM = x_2 - x_1$$

$$BR = BN - RN = BN - AM = y_2 - y_1$$

$\triangle ABR$ is a right angled triangle and by the Pythagorean Theorem,

$$AB^2 = AR^2 + BR^2$$

$$AB = \sqrt{AR^2 + BR^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1. If $A(-3, 3)$, $B(5, 9)$ and $C(-7, 4)$ find the distance between A and B ; B and C.

$$A(-3, 3) \quad B(5, 9)$$

$$AB = \sqrt{\{5-(-3)\}^2 + (9-3)^2} = \sqrt{8^2 + 6^2}$$

$$= \sqrt{64+36} = \sqrt{100} = 10$$

$$B(5, 9) \quad C(-7, 4)$$

$$BC = \sqrt{(-7-5)^2 + (4-9)^2} = \sqrt{(-12)^2 + (-5)^2}$$

$$= \sqrt{144+25} = \sqrt{169} = 13$$

Example 2. A road runs north-south and a river runs east-west. A factory is 5 kms. north of the river and 3 kms. west of the road. Another factory is 4 kms. south of the river and 6 kms. east of the road. Find the length of the telephone line connecting the two factories. Let us take the river as the *x*-axis and the road as the *y*-axis as shown in Fig. 4. The point *O* where they cross is the origin. Let the first factory be *A* and the second be *B*.

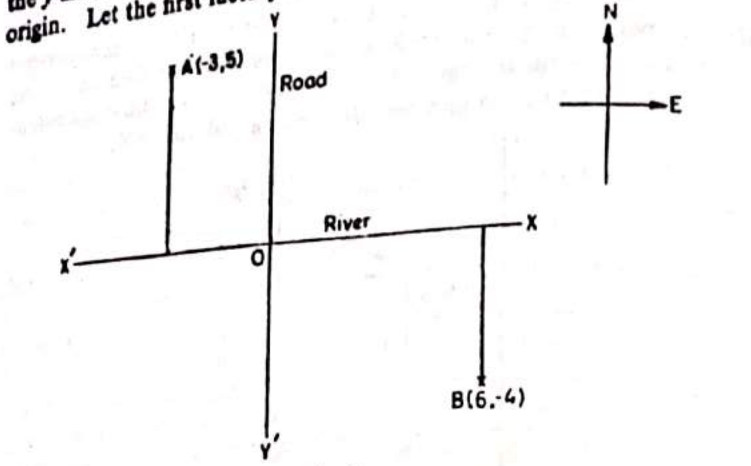


Fig. 4

The factories are represented by the points $A(-3, 5)$ and $B(6, -4)$.

$$AB = \sqrt{\{6-(-3)\}^2+(-4-5)^2}$$
$$= \sqrt{(9)^2+(-9)^2} = \sqrt{81+81}$$
$$= \sqrt{162} = 12{\cdot}73 \text{ kms.}$$

Problem Set 1(a)

(1) Plot the following points :

$A(2, 3)$	$E(0, 8)$	$I(-6, 6)$
$B(-4, -5)$	$F(3, 0)$	$J(7, -3)$
$C(2, -4)$	$G(-2, 0)$	$K(-6, 5)$
$D(-1, 3)$	$H(0, -4)$	$L(-1, 0)$
		$M(3, 3)$

(2) Take the plan of your city and introduce the axes through an important landmark and locate places of importance using co-ordinates.

(3) Referred to some co-ordinate axes an oil well is at the point (3, 4) and oil is sent through pipe lines to the port *A* located at (7, 9) and the port *B* located at (−2, 5) (units are in 100 kilometres) Which port is nearer to the site where oil is drilled ?

(4) The plan of an office room is given. The manager's table is at *P* and one of his assistants is at the table at *Q* (refer Fig. 5). How far will the office boy have to walk to carry a file from *P* to *Q* ? (assume that his path is a straight line). There is a water cooler at *R*. To whom the water cooler is nearer ?

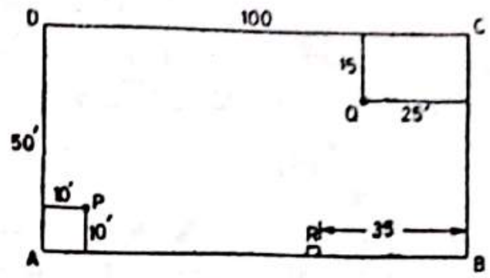


Fig. 5

(5) A tourist car was engaged at Madras whose co-ordinates are (9,8) to go to Kanchipuram whose co-ordinates are (6,0) and Tirupathi co-ordinates (3, −4). Find which is more economical, to go to Tirupathi first or Kanchipuram first if the charges are the same for return trip also.

[*B.B.A. Oct. 1976*]

1·3. Slope of a Straight Line (or Gradient of a Straight Line)

The steepness of a straight line to the *X*-axis is described by a number called the slope of the line. Let us take any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the straight line. The slope is given by the ratio between the vertical separation and horizontal separation of the two points. Thus we have,

$$\text{Slope of the line } AB = \frac{BR}{AR} \text{ (refer Fig. 3)}$$
$$= \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{y_1-y_2}{x_1-x_2}$$
$$= \frac{y \text{ co-ordinate of } A - y \text{ co-ordinate of } B}{x \text{ co-ordinate of } A - x \text{ co-ordinate of } B}$$

y_2-y_1 is called the change in y and is written as Δy (read as delta y) and the change in x is x_2-x_1 written as Δx

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

In practical problems Δy is called the "rise" and Δx the "run". Therefore slope can be defined as the rise per unit of run.

The following figures show that a line (*a*) parallel to the *x*-axis has zero slope, (*b*) perpendicular to the *x*-axis has infinite slope, (*c*) inclined to the right has positive slope and (*d*) inclined to the left has negative slope.

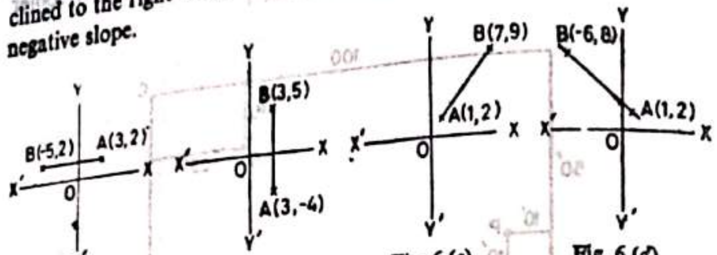


Fig. 6 (*a*) Fig. 6 (*b*) Fig. 6 (*c*) Fig. 6 (*d*)

Fig. 6 (*a*): Slope of $AB = \frac{2-2}{3+5} = 0$

Fig. 6 (*b*): Slope of $AB = \frac{-4-5}{+3-3} = -\frac{9}{0} = \infty$ (Ref. A(4) Appendix 1)

Fig. 6 (*c*): Slope of $AB = \frac{9-2}{7-1} = \frac{7}{6} = +ve$ quantity

Fig. 6 (*d*): Slope of $AB = \frac{2-8}{1+6} = -\frac{6}{7} = -ve$ quantity

Now we just state two important results :

(1) For parallel lines the slopes are equal.

(2) For perpendicular lines the product of the slopes is -1. The reader who is interested in the proof shall refer to problem Nos. 3 and 4 in Problem set 1 (*b*).

Example. Find the slope of the line joining $P(-2, 3)$ and $Q(8, -5)$.

Slope of the line PQ

$$= \frac{y \text{ co-ord. of } Q - y \text{ co-ord. of } P}{x \text{ co-ord. of } Q - x \text{ co-ord. of } P}$$

$$= \frac{-5-3}{8-(-2)} = \frac{-8}{10} = -\frac{4}{5}$$

Problem Set 1 (b)

(1) Taking the points given in problem No. 1 set 1 (*a*) find the slope of the lines

(*a*) AB	(*b*) AD	(*c*) AC	(*d*) MD
(*e*) IJ	(*f*) GE	(*g*) KL	(*h*) CM

comment about their inclination to the *x*-axis.

(2) A mechanic is paid R rupees for H hours. If H increases from H_1 to H_2 then R increases from R_1 to R_2. Draw the line joining the points (H_1, R_1) and (H_2, R_2). What is the slope of this line ? Can you interpret the ratio $\frac{\Delta R}{\Delta H}$?

(3) Consider the following figure and using the idea of similar triangles prove that the slopes of parallel lines are equal.

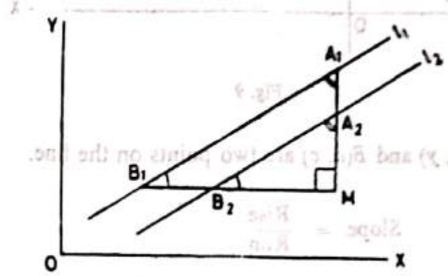


Fig. 7

(4) Two lines l_1 and l_2 are perpendicular (refer Fig. 8). Show that the slope of one line is positive while the slope of the other is negative. Using the figure 8 and results from similar triangles prove that the product of their slopes is -1.

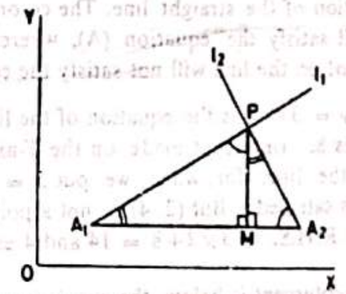


Fig. 8

1·4 Equation of a Straight Line and Applications

In this section few forms of the equation representing a straight line will be discussed.

(*i*) **Slope-intercept Form.** Let a straight line, whose slope is m, cut off an intercept $OB = c$ along the *y*-axis. Let $P(x, y)$ be any point on the line.

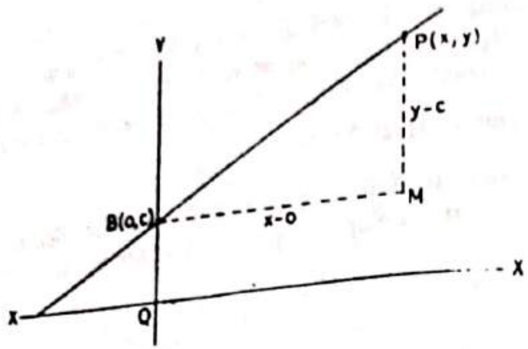


Fig. 9

Now, $P(x, y)$ and $B(o, c)$ are two points on the line.

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

$$m = \frac{y-c}{x-o} = \frac{y-c}{x}$$

$$mx = y-c$$

$$y = mx+c \qquad \text{...(A)}$$

This relation (A) between the co-ordinates (x, y) of any point P is called the equation of the straight line. The co-ordinates of every point on the line will satisfy the equation (A), whereas the co-ordinates of any point not on the line will not satisfy the equation (A).

For example, $y = 3x+8$ is the equation of the line whose slope is 3 and y-intercept is 8. (intercept made on the Y-axis.) The point (1, 11) is a point on the line, for, when we put $x = 1$ and $y = 11$, the above equation is satisfied. But (2, 4) is not a point on the line. For, L.H.S. = 4 and R.H.S. = $3\times2+8 = 14$ and $4 \neq 14$.

Note. If the y-intercept is below the x-axis it will be a negative quantity

i.e., $c < o$, if the y-intercept c lies below the X-axis.

(*ii*) **Point-slope Form.** If the line passes through a given fixed point $A(x_1, y_1)$ and has a given slope m, to find its equation.

Let $P(x, y)$ be any point on the line.

Referring to Fig. 10,

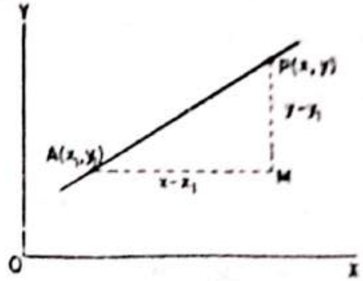


Fig. 10

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{PM}{AM}$$

i.e., $$m = \frac{y-y_1}{x-x_1}$$

$$\therefore \quad y-y_1 = m(x-x_1) \qquad \text{...(B)}$$

(B) gives the equation of the straight line.

(*iii*) **Two Point Form.** Supposing it is given that a line passes through two points, $A(x_1, y_1)$ and $B(x_2, y_2)$.

To write the equation of the straight line.

Then the slope of the line is, $\frac{y_1-y_2}{x_1-x_2}$.

Let $P(x, y)$ be any point on the line. Using result (B), we have

$$y-y_1 = \frac{y_1-y_2}{x_1-x_2}(x-x_1) \qquad \text{...(C)}$$

(C) gives the equation of the straight line joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Example 1. Find the equation of the line whose slope is $\frac{3}{2}$ and which cuts off 3 units along OY.

Here $\quad m = \frac{3}{2}$ and $c = +3$

Equation of the line is,

$$y = \frac{3}{2}x+3 \quad \text{using (A)}$$

$$2y = 3x+6$$

$$3x-2y+6 = 0$$

Example 2. Find the equation of the line passing through the point $(2, -3)$, having the slope $\frac{-5}{7}$.

The equation of the line is,

$$y-(-3) = \frac{-5}{7}(x-2) \quad \text{using (B)}$$

$$y+3 = \frac{-5}{7}(x-2)$$

$$7(y+3) = -5(x-2)$$

$$7y+21 = -5x+10$$

$$5x+7y+11 = 0.$$

Example 3. Find the equation of the line passing through $(2, -3)$ and $(-4, 5)$.

The equation of the line is,

$$y-(-3) = \frac{-3-5}{2-(-4)}(x-2) \quad \text{using (C)}$$

$$y+3 = \frac{-8}{6}(x-2)$$

$$6y+18 = -8x+16$$

$$8x+6y+2 = 0$$

$$4x+3y+1 = 0$$

All the three equations we got above are first degree equations in x and y. In fact any first degree equation in x and y represents a straight line in a plane. That is why any relationship between two variables x and y expressed as a first degree equation in x and y is called the linear relationship.

Example 4. As the number of units manufactured, increases from 2000 to 3000 the total cost of production increases from Rs. 11000 to Rs. 15000. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

When $x = 2000$ $y = 11000$

and when $x = 3000$ $y = 15000$

As the relationship between x and y is linear we have to find the line through (2000, 11000) and (3000, 15000).

The required relationship is,

$$y-11000 = \frac{11000-15000}{2000-3000}(x-2000)$$

i.e. $y-11,000 = 4(x-2000)$

i.e. $4x-y+3000 = 0.$

Example 5. Mr. Ram buys a radio making a payment of Rs. 200 at the time of purchase with the agreement that he will pay at the rate of Rs. 15 for the next 20 months. Find the relationship between the amount (y) he has paid and the number of months (x) since he bought the radio.

From the problem it is evident that when $x=0, y=200$. Thus the y-intercept is 200 *i.e.*, $c = 200$. For a unit change in x the change in y is 15 *i.e.* the vertical change is 15 for a unit horizontal change. Therefore the slope of the line is 15.

Hence the required relationship is,

$$y = 15x+200.$$

1·5. Point of Intersection of Two Lines

Let two lines L_1 and L_2 intersect at a point $P(x_1, y_1)$ *i.e.* (x_1, y_1) is a common point on the lines L_1 and L_2. Then (x_1, y_1) will satisfy both the equations representing the lines L_1 and L_2. In other words x_1 and y_1 give the solution to the two equations which represent L_1 and L_2. So in order to find the point of intersection of two lines, we have to solve their equations for x and y.

Example 6. Find the point of intersection of the lines $5x+2y = 11$ and $x-3y = 9$.

$$5x+2y = 11 \quad \ldots(1)$$

$$x-3y = 9 \quad \ldots(2)$$

Multiplying (1) by 3, $15x+6y = 33$

Multiplying (2) by 2, $2x-6y = 18$

Adding, $17x = 51$

$x = 3.$

Substituting $x = 3$ in (1),

$15+2y = 11$

$2y = -4$

$y = -2.$

The point of intersection is $(3, -2)$.

Example 7. Let two cities located at (2, 1) and (8, 9) be connected by a straight road. Let a third city be at (4, 7). Find the point on the road which should be connected to the third city so that its distance from the road is least. *(B.B.A. Sept. 1973)*

Let the three cities be denoted by A, B, C so that A is the point (2, 1), B (8, 9) and C (4, 7).

Let the perpendicular from C meet the road AB at M. Then M is the point on the road which is the nearest to the city C.

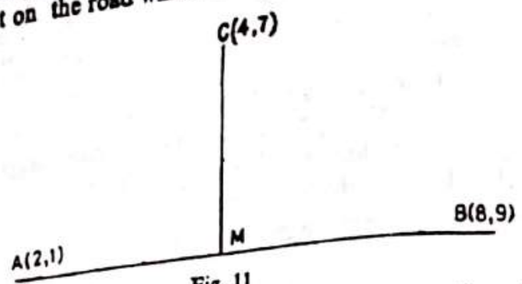


Fig. 11

Since M is the point of intersection of the lines AB and CM we first find the equations of the two lines AB and CM and then solve them.

Slope of the line $AB = \frac{9-1}{8-2} = \frac{8}{6} = \frac{4}{3}$

Equation of AB is,

$$y-1 = \frac{4}{3}(x-2)$$

$$4x-3y-5 = 0 \quad \ldots(1)$$

CM is perpendicular to AB.

$\therefore$ (Slope of CM)×(Slope of AB) $= -1$

i.e., (Slope of CM)$\times\frac{4}{3} = -1$

i.e., Slope of $CM = -\frac{3}{4}$

$\therefore$ Equation of CM is

$$y-7 = -\frac{3}{4}(x-4)$$

$$3x+4y-40 = 0 \quad \ldots(2)$$

Now we solve (1) and (2)

Multiplying (1) by 4,

$$16x-12y-20 = 0$$

Multiplying (2) by 3,

$$9x+12y-120 = 0$$

Adding $\quad 25x-140 = 0$

$$x = \frac{140}{25} = \frac{28}{5}$$

Substituting $x = \frac{28}{5}$ in (1)

$$\frac{112}{5} - 3y - 5 = 0$$

$$3y = \frac{112}{5} - 5 = \frac{87}{5}$$

$$y = \frac{29}{5}$$

$$M \text{ is } \left(\frac{28}{5}, \frac{29}{5}\right)$$

Problem Set 1 (c)

(1) Find the equation of the following lines whose slopes and y intercepts are given.

(*a*) slope $-\frac{1}{2}$, $\quad$ y-intercept $\frac{4}{3}$

(*b*) slope $\frac{3}{4}$, $\quad$ y-intercept 7

(*c*) slope 3. $\quad$ y-intercept $-\frac{2}{7}$

(2) Find the equation of the line which passes through a given point and having the given slope

(*a*) (5, 4), Slope 3.

(*b*) (−3, 2), Slope −2.

(*c*) (2, −5), Slope $-\frac{3}{4}$

(*d*) (−4, −6), Slope $\frac{5}{3}$

(*e*) (2, −3), Slope $\frac{-4}{3}$ [*B.B.A. Oct. 76*]

(3) Find the equation of the line which passes through, the pair of points given below.

(*a*) (2, 4) and (3, 4)

(*b*) (−3, 1) and (2, −1)

(*c*) (−4, −4) and (5, 3)

(4) Find the point of intersection of the following lines

(*a*) $x+3y-5 = 0$ $\quad$ $x-2y+5 = 0$

(*b*) $3x+4y-13 = 0$ $\quad$ $2x-7y+1 = 0$

(*c*) $x+3y+2 = 0$ $\quad$ $2x-y-3 = 0$

(*d*) $7x-y-3 = 0$ $\quad$ $2x-5y-15 = 0$

(5) A man has to pay Rs. 5,000 initially and then Rs. 300 for every month for 3 years, for a small flat he has bought. Find the relationship between the amount (y) he has paid and the number of months (x) since he bought the flat, assuming the relationship to be linear.

(6) As the number of manufactured units increase from 300 to 600 the total cost of production increases from Rs. 1500 to Rs. 2700. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

(7) If taxi fare (y) is Rs. 2 minimum plus 75 paise per kilometre, write the equation connecting the fare and the kilometres travelled (x).

(8) Two cities are located at A (1, 2) and B (9, 14). Find the point on the road connecting A and B which is the nearest point to the city C located at (6, 3).

(9) Solve the problem (8) if

(i) A is (1, 2), B is (9, 8) and C is (2, 1)

(ii) A is (5, 0), B is (−5, 0) and C is (0, 100)

(iii) A is (2, 5), B is (3, 5) and C is (10, 0)

(10) A plant is at A with co-ordinates (0, 2). Its produce is to be trucked to each of the warehouses B and C with co-ordinates (3, 6) and (6, 2) respectively. Find which is more economical : to touch B first or to go to C first if only available roads are between any pair of points. If there were a fourth road connecting B to the nearest point D on the road AC, would it be still better to follow $ADBC$. [B.B.A. April 1973]

(11) Find the equation of the straight line which makes intercepts a and $3a$ on the co-ordinate axes and passes through the point (3, −4). [B.B.A. April 1976]

1.6. Interpretations

(i) **Cost-Output.** A useful interpretation to the slope concept may be given as follows. Let us assume that to produce Q units of a product, C is the total cost of production and that the relation between C and Q is linear. This relation is exhibited by a straight line. Graphically AS and BT represent the total cost when OS and OT units are produced respectively. OC represents the cost when no (i.e., zero) unit is produced. OC is called the fixed cost which exists even when there is no production.

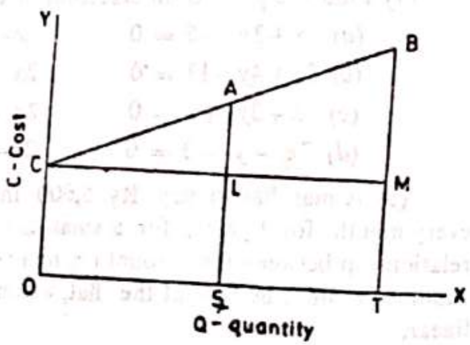


Fig. 12

From Figure 12, $MT = LS = CO$.

When OT units are produced the total cost is BT.

The total cost $(BT) = MT + MB$.

$= \text{Fixed Cost} + MB$.

We call MB the variable cost when OT units are produced. Likewise LA is the variable cost when OS units are produced. Thus at any stage the total cost is the sum of the fixed cost and variable cost.

The variable cost per unit

$$= \frac{\text{Variable cost}}{\text{No of units made}}$$

$$= \frac{AL}{OS}$$

$$= \frac{AL}{CL}$$

= slope of the line

For straight line the slope is constant. Hence we conclude that "if costs are linearly related to output the variable cost per unit is constant and conversely."

The change in total cost when one more unit is produced is called the **Marginal Cost.** It is the change in y for a unit change in x and therefore it is the slope of the line. Thus, when the cost is linearly related to the output the marginal cost is a constant.

But the **Average Cost,** which is defined as the total cost over the number of units produced, is not constant. Consider for example $y = 10x + 1000$ where y represents the total cost and x the number of units produced. Here Rs. 1000 represents the fixed cost and the marginal cost is 10. When 10 units are produced the total cost is Rs. 1100 and hence the average cost is Rs. 1100/10 = 110. When 20 units are produced the total cost is Rs. 1200 and hence the average cost is Rs. 1200/20 = Rs. 60. This shows that the average cost is not a constant and it declines, as production increases, as a consequence of the spreading of the fixed cost over larger number of units.

Example. The total factory cost (y) of making x units of a product is given by $y = 5x + 300$, and 75 units are made. Find (i) the fixed cost (ii) the variable cost (iii) the total cost (iv) the variable cost per unit (v) average cost per unit (vi) the marginal cost.

(i) Fixed cost is Rs. 300.

(ii) When x units are produced variable cost is $5x$. So here the variable cost = Rs. 5×75 = Rs. 375.

(*iii*) The total cost $= 5\times75+300 =$ Rs. 675.

(*iv*) The variable cost per unit = slope of the line = Rs. 5.

(*v*) Average cost $= \frac{\text{Total cost}}{\text{No. of units}} = \frac{675}{75} =$ Rs. 9.

(*vi*) The marginal cost = slope of the line = Rs. 5.

(*ii*) **Demand and Supply Curves (Linear).** For the sake of simplicity we use linear equations for both demand and supply curves. It is also quite reasonable to do so in certain situations.

Linear Demand Curve. Usually the slope of a demand curve is negative, that is, as price increases demand decreases and as price decreases demand increases (Fig. 13).

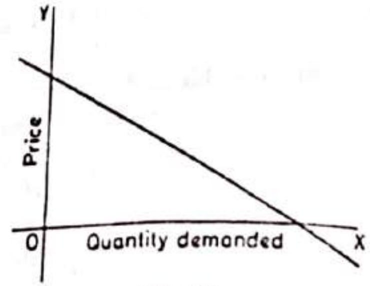


Fig. 13

Example. 15 radios are sold when the price is Rs. 400 and 25 radios are sold when the price is Rs. 350. What is the equation of the demand curve assuming it to be linear ?

Denote the demand by x and the price by y.

When, $x = 15$ $y = 400$

and when $x = 25$ $y = 350$

Thus the demand curve is the straight line through the points (15,400) and (25,350).

$$\therefore \quad y-400 = \frac{400-350}{15-25}(x-15)$$

$$y-400 = -5(x-15)$$

$$y-400 = -5x+75$$

$5x+y-475 = 0$ is the required equation.

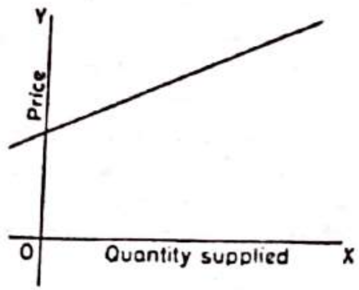


Fig. 14

Linear Supply Curve. Usually the slope of the supply curve is positive as the price increases supply increases and as the price decreases supply decreases. Ref. Fig. 14.

Example. **When the price is Rs. 50, 60 cameras of a particular type are available and when the price is Rs. 80, 140 cameras of the same type are available in the market. Determine the supply curve.**

Let x denote the quantity supplied and y the price.

When $x = 60$ $y = 50$

and when $x = 140$ $y = 80$

Thus the supply curve is the straight line through the points (60, 50) and (140, 80).

$$\therefore \quad y-50 = \frac{50-80}{60-140}(x-60)$$

$$y-50 = \frac{3}{8}(x-60)$$

$$8y-400 = 3x-180$$

$3x-8y+220 = 0$ is the required equation.

Market Equilibrium. Market equilibrium is said to occur at the point (price) at which the quantity of a commodity demanded is equal to the quantity supplied. Thus the equilibrium amount and equilibrium price correspond to the co-ordinates of the point of intersection of the demand and supply curves.

The equilibrium is meaningless if the point of intersection E lies outside the I quadrant.

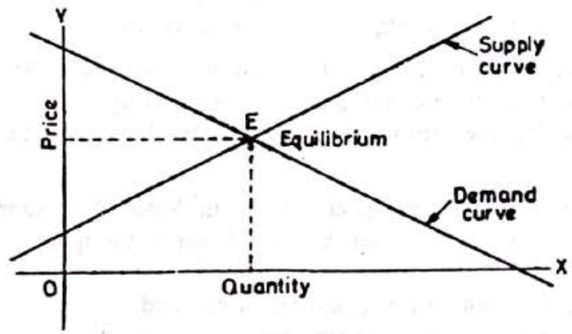


Fig. 15

Problem Set 1 (d)

(1) If the total cost (y) of making x units is given by

$$y = 4x+50$$

and if 60 units are made. Find

(*a*) The fixed cost.

(*b*) The variable cost.

(*c*) The total cost.

(*d*) Variable cost per unit.

(*e*) Average cost per unit.

(*f*) The marginal cost

(2) The total cost (y) of making x units is given by,

$$y = 2x+30$$ and 50 units are made.

Answer the subdivisions (a), (b), (c), (d), (e), and (f) of problem number 1 above.

(3) When the price of an electric heater is Rs. 30, 100 people will buy and when the price is Rs. 25, 130 people will buy. Obtain the equation of the demand curve.

(4) At a price of Rs. 7 per bottle a company will supply 6000 squash bottles every month and at Rs. 5 per bottle it will supply 4000 bottles. Find the supply curve.

(5) Identify which of the following equations can represent a demand curve and which can represent a supply curve. Graph the curves. Determine the equilibrium point from the graph and verify it algebraically. (x quantity, y price)

(a) $y = 9-2x$, $\quad y = \frac{3}{2}x+1$

(b) $y = 15-3x$, $\quad x = 2y-3$

(c) $x+y = 5$, $\quad 2x-y = 5{\cdot}5$

(d) $2y+3x = 10$, $\quad x = 4y-6$

(6) A firm found out that its customers will buy 15% more of its product if the price of the product is reduced by Rs. 5. When the price is Rs. 25, the firm is selling 1000 units. Determine the demand curve.

(7) Now there are special cases in demand and supply. The various situations are given below. Make out a figure to illustrate them.

(1) Constant price regardless of demand.

(2) Constant demand regardless of price.

(3) Constant price regardless of supply.

(4) Constant supply regardless of price.

1·7. Break-even analysis

Let us assume that linear relationship exists between the sales revenue and expenses. Let y represent the expense incurred when the sales is x. Let the relationship be, $y = mx+c$.

Let $\quad y = 0{\cdot}7x+10{,}800 \quad$ ...(1)

The slope of the line is 0·7. Rs. 0·70 is the variable expense per rupee of sales, for, 0·7 is the change in y for a unit change in x. Rs. 10,800 is the fixed expense.

The **Break Even Point** is the volume of sales at which sales equals expenses, that is, the point at which the company experiences neither a loss nor a profit. Let the break-even level of sales be represented by X_e

At this stage

$$x = y = X_e \quad ...(2)$$

Using (2) in (1)

$$X_e = 0{\cdot}7\,X_e + 10{,}800$$
$$0{\cdot}3\,X_e = 10{,}800$$
$$X_e = 36{,}000$$

Thus when the revenue from sales is Rs. 36,000 the expenses is also Rs. 36,000 (verify) and the company has neither profit nor loss.

In general, if the relation is $y = mx+c$, then the break-even level X_e will be given by

$$X_e = mX_e + c$$
$$X_e(1-m) = c$$
$$X_e = \frac{c}{1-m}$$

$$\text{Break-even sales} = \frac{\text{Fixed Expenses}}{1-\text{Variable expense per rupee of sales}}$$

Example. A company estimates that when its sales is Rs. 60,000, its variable expense will be Rs. 30,000 for a fixed expense of Rs. 10,000. Find the break-even point. What is the profit when the sales is Rs. 50,000 ?

The variable expense per rupee of sales

$$= \frac{30{,}000}{60{,}000} = {\cdot}5$$

Now the slope of the line representing the relation between the sales (x) and expenses (y) is 0·5.

$$\therefore \quad y = 0{\cdot}5x+10{,}000$$

is the required relation.

At the break-even level, $y = x = X_e$

$$X_e = 0{\cdot}5\,X_e + 10{,}000$$
$$0{\cdot}5\,X_e = 10{,}000$$
$$X_e = 20{,}000$$

This shows that only when the company's sales exceed Rs. 20,000 there will be profit for the company.

When the sales is Rs. 50,000,

$$x = 50,000$$
$$y = 0{\cdot}5 \times 50,000 + 10,000$$
$$= 25,000 + 10,000$$
$$= 35,000$$

Profit = Revenue from Sales − Expenses
= Rs. 50,000 − Rs. 35,000
= 15,000.

The following chart gives a clear—cut idea of the situation.

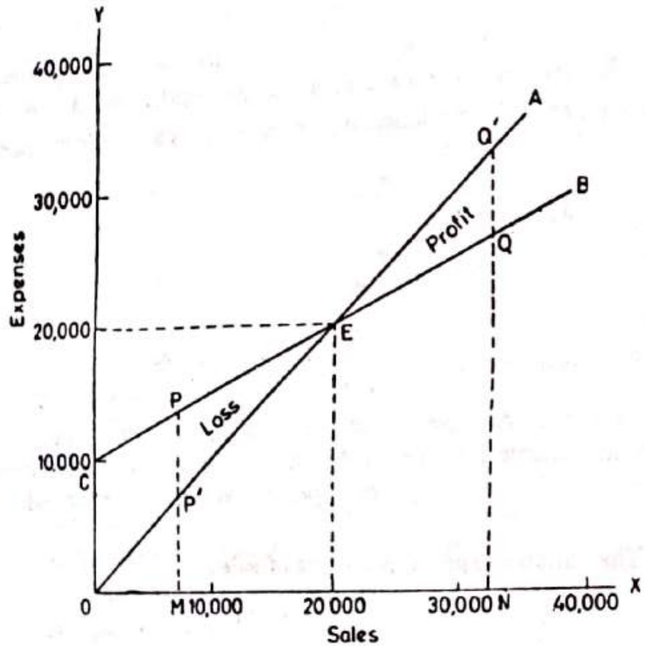


Fig. 16

CB represents the line relating sales and expense *viz.* the line,

$$y = 0{\cdot}5\,x + 10,000 \qquad \text{...(1)}$$

Now we take the line *OA* whose slope is 1, that is, for a unit change in x the change in y is one unit. The line passes through the origin. Its equation will be,

$$y = x \qquad \text{...(2)}$$

For any point on line (2), the x and y co-ordinates will be equal. Suppose this line were to describe the state of affairs of the company, what we conclude is that the sales revenue and expenses are equal. This will mean that the company works on no profit and no loss. Usually such a thing never happens in reality.

Let E be the point of intersection of the lines (1) and (2). Then the co-ordinates of E which are equal wlll give the break-even point.

Let P and P' be corresponding points on the lines (1) and (2) to the left of E.

Then P is (OM, MP) and P' is (OM, MP')

Since P' lies on the line $y = x$ we get $OM = MP'$

But $\qquad MP' < MP$

$\therefore \qquad OM < MP$

i.e. the revenue from sales is less than the expenses.

$\therefore$ The company is working on loss.

Let Q, Q' be two corresponding points on the lines (1) and (2) to the right of E.

Then Q is (ON, NQ) and Q' is (ON, NQ')

Since Q' lies on the line $y = x$ we have $ON = NQ'$

Now $\qquad NQ' > NQ$

$\therefore \qquad ON > NQ$

i.e. the revenue from sales is greater than the expenses. Therefore, the company is working on profit.

So it is clear that a company should take steps to increase the sales beyond the break-even level of sales to realise profit.

Problem Set 1 (e)

(1) A company expects fixed costs to be Rs. 30,000 and variable cost to be Rs. 42,000 when the sales will be Rs. 60,000

(*a*) Write down the equation relating sales and expenses.

(*b*) Find the break-even point.

(*c*) What will be the profit when the sales is Rs. 1,20,000

(*d*) Draw the break-even chart.

(2) A company expects fixed costs of Rs. 37,500 and variable cost of Rs. 50,000 on sales of Rs. 80,000.

(*a*) Write down the equation relating the cost and sales.

(*b*) Find the break-even point.

and work as above.

$$y-20 = x^2-8x$$
$$y-20+16 = x^2-8x+16$$
$$y-4 = (x-4)^2$$

$\therefore$ The vertex is (4, 4).

Alternatively, using the above result, (B) we have

$$x = -\frac{b}{2a} = -\frac{(-8)}{2\times 1} = 4$$
$$y = -\frac{(b^2-4ac)}{4a} = -\frac{(64-4\times 1\cdot 20)}{4\cdot 1} = 4$$

$\therefore$ The vertex is (4, 4).

Applications. We have considered the case when supply curve and demand curves were straight lines. We take the case when the demand and supply curves are parabolas.

Example. The demand and supply curves are given by

$$y = 10-3x^2 \quad \text{and} \quad y = 4+x^2+2x$$

(y represents the price and x the quantity). Find the equilibrium price and quantity.

Demand curve, $y = 10-3x^2$ ...(1)

Supply curve, $y = 4+x^2+2x$ ...(2)

At the equilibrium level the price for supply and demand are equal. Equating (1) and (2),

$$10-3x^2 = 4+x^2+2x$$
$$4x^2+2x-6 = 0$$
$$2x^2+x-3 = 0$$
$$x = \frac{-1\pm\sqrt{1^2-4(2)(-3)}}{2\times 2}$$

(Refer D (2) Appendix I)

$$= \frac{-1\pm\sqrt{25}}{4}$$
$$= \frac{-1\pm 5}{4}$$
$$= 1 \text{ or } -\frac{3}{2}$$

$x = -\frac{3}{2}$ is inadmissible.

Substituting $x = 1$ in (1) we get $y = 7$. Then at the equilibrium level the quantity supplied and demanded is 1 unit and the price is 7 units.

Problem Set I (f)

(1) Find the vertex of the following parabolas :

(*i*) $y = x^2-5$

(*ii*) $y = -2x^2+10$

(*iii*) $y = x^2-2x+6$

(*iv*) $y = -x^2+6x-7$

(*v*) $y = 3x^2-12x+5$

(2) Find the equilibrium price and quantity for the following :

(*i*) Supply curve ; $y = x^2+5x+1$
Demand curve ; $y = 9-2x^2$

(*ii*) Supply curve ; $y = 8x+7$
Demand curve ; $y = 27-x^2$

(*iii*) Supply curve ; $4y = x^2+2x+5$
Demand curve ; $y = -2x+20$

(3) Plot the graph of the quadratic function

$$y = 3x^2+4x+3$$

[*B.B.A. Sept. 75*]

UNIT-2

2

Sets, Relations and Functions

2·1. Basic Concepts.

A collection of objects is called a **Set.**

Examples :

(*i*) The board of directors of a company.

(*ii*) All equipment in a firm

(*iii*) Consumers of a product

(*iv*) Employees in a firm

(*v*) All even numbers.

Given an object we should be able to say whether the object belongs to the set or not. In listing the objects belonging to a set no object is repeated in the listing after it has been recorded once. The set of price quotations of a common stock during one day is observed as follows :

Rs. 11.00, Rs. 11.50, Rs. 12.50, Rs. 13.80, Rs. 11.00, Rs. 12.50 is a set not represented by six numbers but by four distinct numbers.

11.00, 11.50. 12.50, 13.80.

An object which belongs to or is a member of a set is called an **Element** of the set. Capital letters are used to denote sets while small letters are used to denote the elements of the set, unless otherwise stated. If x is an element of a set P it is symbolically represented by "$x \in P$" where '$\in$' stands for the words "belongs to". $\notin$ denotes "does not belong to". If y is not an element of a set P then we write $y \notin P$.

A set is designated either by listing all the elements within braces { } or by giving the property which the elements should satisfy to belong to the set.

Example 1. The set of all odd numbers less than 7, denoted by P, can be given either as

$$P = \{1, 3, 5\}$$

by listing all the numbers satisfying the condition, or as,

$$P = \{x : x \text{ is an odd number less than } 7\}$$

stands for "such that". Sometimes a bar : is used to denote:such that"

$\{x : x$ is an odd number less than $7\}$ is read as the collection or set of all elements x such that x is an odd number less than 7.

Example 2. Suppose Messrs. Ramesh, Bapu, John, Gafoor, Sekar constitute the Board of directors of Southern Transport company. They form a set A which we write as,

$A =$ {Mr. Ramesh, Mr. Bapu, Mr. John, Mr. Gafoor, Mr. Sekar}.

or $A = \{x : x$ is a director of the Southern transport company$\}$,

The latter representation is very convenient when the set has many elements. For example if we take the set of all men in Madras city who use Star toothpaste, it will be quite unwieldy to write the names of all the persons within the braces. Whereas we can very briefly write the set as,

$S = \{x : x$ is a person living in Madras city who uses Star Tooth paste$\}$

A set having finite number of elements is called a **Finite Set.** A set which is not finite is called an **Infinite Set.** Examples (*i*), (*ii*), (*iii*), (*iv*) are finite sets, while (*v*) serves as an example for an infinite set.

A set which has no elements is called an **Empty Set** or **Null Set** or **Void Set** and is denoted by ϕ. Some authors use "O" to denote a null set.

Suppose in Oriental Mills Ltd., the pay scale of senior executives is Rs. 2,000-200-3,000-300-4,500. If we define a set P as follows,

$P = \{x : x$ is a senior executive In Oriental Mills limited drawing Rs. 1,900 p.m.$\}$

This set will not have any element and therefore is a null set. Thus,

$\{x : x$ is a senior executive in Oriental Mills Ltd. drawing Rs. 1,900 p.m.$\} = \phi$

In any analysis of a particular situation the fixed collection of all elements needed for the analysis is defined as the **Universal**

Set denoted by U (or I). Then a particular set is specified by referring to the universal set.

Suppose we want to study about a problem connected with certain number of workers of an industry. Then the collection of all workers in the industry will naturally be the universal set.

2·2 Subset

Definition. A is said to be a **Subset** of the set B if every element of A is also an element of B.

If A is a subset of B, we say A is contained in B and write it symbolically as $A \subseteq B$. "$\subseteq$" stands for the words "contained in" or "is a subset of".

Example. (*i*) Let I be the set of integers and E the set of even numbers. Then $E \subseteq I$.

(*ii*) The set of all consumers of a product in Bombay is a subset of all consumers of the product in India.

Set Equality

Two sets A and B are said to be equal if A has the same elements as in B and *vice versa*. That is, two sets A and B are said to be equal if and only if $A \subseteq B$ and $B \subseteq A$. Consider the sets

$$A = \{1, 2, 3, 4\} \text{ and } B = \{1, 2, 3, 5\}.$$

$$A \neq B \text{ and neither } A \subseteq B \text{ nor } B \subseteq A.$$

Note : (*i*) If two sets are unequal one set may or may not be the subset of the other

(*ii*) A set is trivially a subset of itself.

(*iii*) ϕ is a subset of any set.

Number of Subsets of a given finite set. Now let us examine the number of subsets which can be formed out of a given finite set having n elements.

Consider first the set having no element. Only one subset can be formed namely ϕ which is the same as the original set.

Now we take a set containing only one element, say, a *i.e.* $\{a\}$. Two subsets are possible, the null set and the one element set,

$$i.e.\ \phi, \{a\}$$

Thus the number of subsets that can be formed from a set containing one element is $2 = 2^1$.

Consider a set with two elements, say, the set $\{a, b\}$. The subsets are, ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$. The number of such subsets is $4 = 2^2$.

Let $\{a, b, c\}$ be a set with three elements a, b and c. The subsets are,

$$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\ \{c, a\}, \{a, b, c\}$$

The number of subsets is $8 = 2^3$.

Proceeding like this, by induction, we get the result that,

"A set with n elements has 2^n subsets". This includes the null set and the given set.

Problem Set 2 (a)

(1) Find all subsets of the set A where $A = \{a, b, c, d, e\}$.

(2) Find the total number of (*i*) two element subsets (*ii*) three element subsets of the set $\{x, y, z, w, t\}$

(3) Let a set A have 15 members. Find the total number of subsets having at least two members which can be formed out of A. *(M.B.A. 1973)*

(4) Find the total number of subsets of the set $\{1, 2, 3, 4, 5\}$ which contain

(*i*) odd number of elements
(*ii*) even number of elements
(*iii*) odd numbers only
(*iv*) even numbers only

(5) Consider the set E of all employees in a firm and the subset of persons described by the following definition : two persons $x, y \in E$ are in the same subset if they do the same type of job. What can you say about the different subsets thus formed.

(6) If subsets are formed out of A in problem (5) by defining that any two persons, $x, y \in E$ are in the same subset if they get the same salary. What can you say about the different subsets thus formed ?

(7) A company's research budget is Rs. 60,000. It is considering the allocation of this budget to four research projects with the following requirements for each :

Project	Requirement
1	15,000
2	25,000
3	35.000
4	10,000

Find the total costs of the 16 possible subsets of projects and select the subsets that do not exceed in cost the planned budget.

block the measure by his single vote alone. We say that such a member has got VETO power. There may be a member whose influence is such that any winning coalition of which he is a member will still be a winning coalition without him. Such a member is called a **Powerless member.** Thus, in short, a member x is a dictator if and only if $\{x\}$ is a winning coalition. A member x has veto power if and only if $\{x\}$ is a blocking coalition.

Problem Set 2 (b)

1. Find the winning, losing and blocking coalitions in the Security Council of the U.N.O.

2. Prove that any two minimal winning coalitions have atleast two members in common.

3. Prove that if a committee has a dictator as a member the remaining members are powerless.

4. A company has issued 10,000 shares of common stock and each share has one vote. How many shares must a person have

(*i*) to be a dictator

(*ii*) to have a veto

(*iii*) to be a dictator or to have a veto if the company requires two-thirds majority to carry an issue.

5. Let the universal set be $U = \{a, b, c, d\}$, where a and b have each one vote, c has two votes and d three votes. Suppose that it takes 5 votes to carry a measure. List the winning, loosing and blocking coalitions.

6. Find the set of all sub-sets of U where $U = \{1, 2, 3, 4, 5, 6\}$.

7. A factory committee consists of two management members and two union members. For a proposal to be passed on from the committee to the stock-holders for consideration, a simple majority of the members must vote for the proposal. Each member has one vote.

(*a*) How would you describe in set terms a group that could veto any proposal brought before it ? How many such groups are there in the present situation ?

(*b*) If one member has the power to veto any decision by his no vote, list or define in set terms the groups that could veto a proposal. How many groups would there be ?

2·4. Operations On Sets—Applications

We have seen that one way of generating new sets from the elements of a given universal set is by subset forming. There is another way of forming new sets from given sets using certain operations on sets, which will be given here.

There is a way of visualising set operations and performing calculations with sets by means of a diagram. A rectangle is used to represent the universal set U. All the members of the universal set will be represented by points in it. A circle inside the rectangle will represent a set which is got from U, the members of the set being the points inside the circle. To every sub-set of U we can associate a circle. Such a diagram is called the Venn Diagram (refer figure 1).

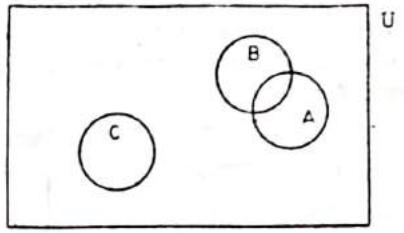


Fig. 1

Union. Let A and B be any two sets. The union of A and B, denoted by $A \cup B$, is the set of all elements which are members of either A or B or both.

i.e., $A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}$

The shaded area in figure 2 represents $A \cup B$.

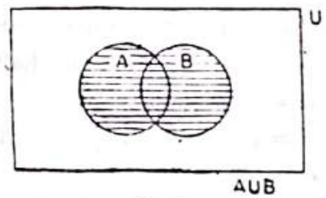


Fig. 2

Examples : (*i*) $A = \{2, 7, 3\}$, $B = \{4, 5\}$, $A \cup B = \{2, 7, 3, 4, 5\}$

(*ii*) $A = \{2, 9, 10\}$, $B = \{9, 10, \frac{1}{2}, -5\}$

$A \cup B = \{2, 9, 10, \frac{1}{2}, -5\}$

(*iii*) $A = \{x, y\}$, $B = \{y\}$

$A \cup B = \{x, y\} = A$

Notice that $B \subseteq A$. Infact, $A \cup B = A$ if $B \subseteq A$

(iv) $A = \{$Krishnan, Prasad, Sunil, Joseph$\}$

$B = \{$Prasad, Sunil, Ahmed, Ganesh$\}$

$A \cup B = \{$Krishnan, Prasad, Sunil, Joseph, Ahmed, Ganesh$\}$

Facts (i) For any set A, $A \cup A = A$

(ii) $A \cup U = U$ and $A \cup \phi = A$

(Verify these using Venn Diagram).

Intersection. Let A and B be any two sets. Then the intersection of A and B, denoted by $A \cap B$, is the set of all elements which are members of both A and B.

i.e., $A \cap B = \{x : x \in A \text{ and } x \in B\}$. $A \cap B$ gives the collection of all elements common to both A and B. The shaded area in figure 3 represents $A \cap B$.

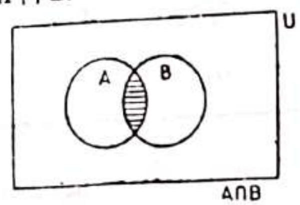


Fig. 3

In some books $A \cap B$ is denoted by AB. The intersection of A and B, is also called the product of A and B.

Examples : (i) $A = \{1, 7, 3\}$, $B = \{2, 3, 5\}$

$A \cap B = \{3\}$

(ii) $A = \{$Krishnan, Suresh, Goel, Pratap$\}$

$B = \{$Suresh, Mukerjee, Balbir, Goel, Peter$\}$

$A \cap B = \{$Suresh, Goel$\}$

(iii) $A = \{-1, 3, -5, 8\}$ $\quad B = \{6, 7, -2, -3\}$

$A \cap B = \phi$

Two non-empty sets A and B are called disjoint if $A \cap B = \phi$.

Facts : (i) $A \cap A = A$ and $A \cap U = A$

(ii) If $B \subseteq A$, then $A \cap B = B$

(iii) $A \cap \phi = \phi$

Complement. Let A be any subset of some universal set U. Then the complement of A denoted by A' is the set of all elements of U which do not belong to A.

i.e., $\quad A' = \{x : x \in U \text{ and } x \notin A\}$

The shaded portion in figure 4 represents A'.

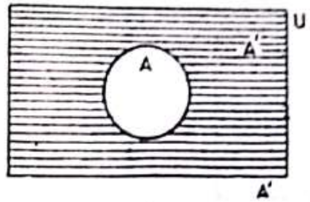


Fig. 4

Some authors use $\bar{A}$ or $\sim A$ to denote the complement.

Examples : (i) $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{3, 5, 9, 10\}$

Then $\quad A' = \{1, 2, 4, 6, 7, 8\}$

(ii) Let N be the set of all positive integers and E, the set of all positive even numbers. Then E' is the set of all positive odd numbers.

(iii) Let $\quad U = \{x$: x is an administrative officer in a firm$\}$

and $\quad A = \{$Mr. Om Prakash, Mr. Raman, Mr. Reddy$\}$

and $\quad A \subseteq U$.

Then $\quad A' = \{x$: $x \in U$ but x is not Mr. Om Prakash, Mr. Raman and Mr. Reddy$\}$

(iv) If $\quad U = \{x$: x is a manufacturing company registered in Tamil Nadu$\}$

and $\quad A = \{x$: x is a manufacturing company registered in Tamil Nadu producing items for export$\}$

then $\quad A' = \{x$: x is a manufacturing company registered in Tamil Nadu producing items for home consumption only$\}$

Note that,

$$A \cup A' = U,\ A \cap A' = \phi,\ (A')' = A$$

De Morgan's Laws :

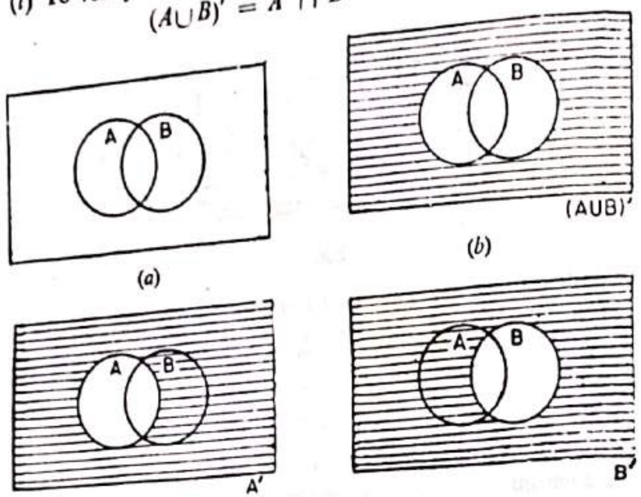
Verify that,

(i) $(A \cup B)' = A' \cap B'$

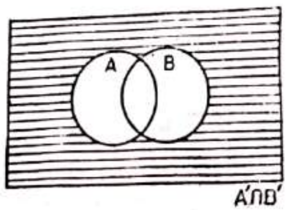
(ii) $(A \cap B)' = A' \cup B'$ using Venn diagram.

(*i*) To verify that

$$(A \cup B)' = A' \cap B'$$



(*a*) (*b*) (*c*) (*d*)



Using figures 5 (*c*) and 5 (*d*)

Fig. 5 (*e*)

Comparing 5(*e*) with 5(*b*) result (1) follows

(*ii*) $(A \cap B)' = A' \cup B'$ can be verified in the same way.

(*i*) and (*ii*) are called De Morgan's Laws. These can be extended to any number of sets.

Set Difference. Let A and B be any two sets (subset of some universal set U). The difference $A-B$, of two sets A and B is the set whose elements are those elements in A which are not in B *i.e.*, $A-B=\{x:x \in A \text{ and } x \notin B\}$. $A-B$ is read as A minus B or A difference B. $A-B$ is also known as the relative complement of B in A. The shaded portion in figure 6 represents $A-B$.

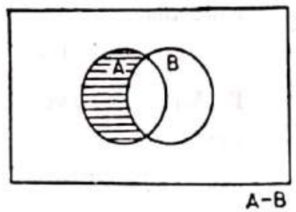


Fig. 6

Example : Let $U = \{1, 2, 3, 4\}$ $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

Then $A-B = \{1\}$ and $B-A = \{4\}$

Now, $A \cup B = \{1, 2, 3, 4\}$ and $A \cap B = \{2, 3\}$

Then $(A \cup B)-(A \cap B) = \{1, 4\}$ ...(1)

$A \cap B' = \{1\} = A-B$

This result, $A-B = A \cap B'$ is always true

Similarly $B-A = B \cap A'$ ...(2)

Now $(A-B) \cup (B-A) = \{1, 4\}$ ...(3)

Comparing (1) and (3) we get,

$$(A-B) \cup (B-A) = (A \cup B)-(A \cap B)$$

Using the results (2) and (3) we get,

$$(A-B) \cup (B-A) = (A \cap B') \cup (B \cap A')$$

Definition. The symmetric difference of two sets A and B, denoted by $A \triangle B$ is defined as

$$(A \cup B)-(A \cap B)$$

The set $(A-B) \cup (B-A)$ is clearly the symmetric difference of the sets A and B. The shaded area in figure 7 represents $A \triangle B$.

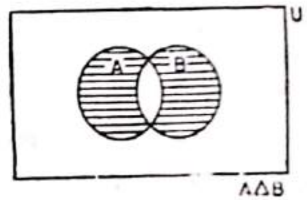


Fig. 7

Laws of set operations

	Union	*Intersection*
1. (*a*)	$A \cup \phi = A$	(*b*) $A \cap U = A$
2. (*a*)	$A \cup B = B \cup A$	(*b*) $A \cap B = B \cap A$
3. (*a*)	$A \cup (B \cap C) = (A \cup B) \cup C$	(*b*) $A \cap (B \cap C) = (A \cap B) \cap C$
4. (*a*)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. (*a*)	$A \cup U = U$	(*b*) $A \cap \phi = \phi$
6. (*a*)	$A \cup A = A$	(*b*) $A \cap A = A$

Complementation

7. (*a*)	$A \cup A' = U$	(*b*) $A \cap A' = \phi$
8. (*a*)	$(A')' = A$	

9. (*a*) $(A \cup B)' = A' \cap B'$ (*b*) $(A \cap B)' = A' \cup B'$

10. (*a*) $U' = \phi$ (*b*) $\phi' = U$

Set Difference :

11. $A - B = A \cap B'$

12. $A \triangle B = (A - B) \cup (B - A)$

$= (A \cap B') \cup (A' \cap B)$

$= (A \cup B) - (A \cap B)$

An Important Note : The operations $\cup$ and $\cap$ are logically equivalent to "or" (inclusive or) and "and" respectively.

EXAMPLES :

Example 1. In a market survey people are to be classified according to whether they smoke cigarettes, pipes, cigars or do not smoke at all. Let *A*, *B*, *C* be the set of persons who smoke cigarettes, pipes and cigars respectively. Describe the persons who belong to the following sets.

(*a*) $A \cap B$ (*b*) $A \cap C$ (*c*) $B \cap C$

(*d*) $A \cup B \cup C$ (*e*) $A \cap B \cap C$ (*f*) $A' \cap B' \cap C'$

(*g*) $A' \cup B' \cup C'$ (*h*) $A \cap (B' \cup C')$ (*i*) $A \cap (B \cap C')$

(*j*) $A \cup (B \cap C')$

(*a*) Since $A \cap B = \{x : x \in A \text{ and } B\}$, $A \cap B$ is the set of all persons who smoke both cigarettes and pipes.

(*b*) , (*c*) Similar to (*a*)

(*d*) $A \cup B \cup C$ is the set of all persons who smoke cigarettes, or pipes or cigars or any two of them or all the three *i.e.*, the set of all persons who smoke at least one brand.

(*e*) $A \cap B \cap C$ is the set of all persons who belong to the sets *A*, *B* and *C*, and, therefore the set represents the set of persons who smoke all the three *i.e.*, cigarettes, pipes and cigars.

(*f*) $A' \cap B' \cap C'$

By De Morgan's Law,

$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

$A' \cap B' \cap C'$ is the complement of the set $A \cup B \cup C$ *i.e.*, $A' \cap B' \cap C'$ is the set of all persons who do not belong to $A \cup B \cup C$ *i.e.*, the set of all persons who do not smoke.

(*g*) $A' \cup B' \cup C'$ is the set of all persons who do not belong to all the three sets *A*, *B*, *C* *i.e.*, the set of persons who do not smoke all the three varieties. This is the set of persons who do not smoke at all or who smoke any one only or two at the most *i.e.*, the set of all persons who smoke at the most two of the three *viz.* cigarettes, pipes and cigars.

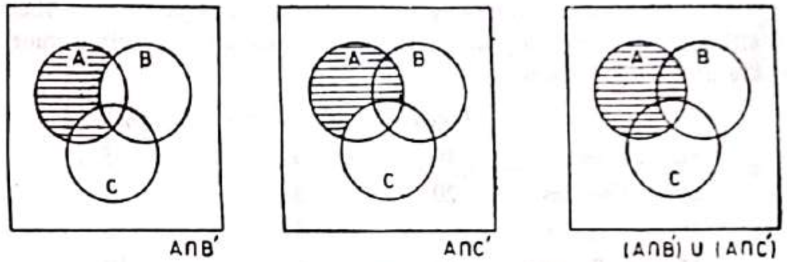
(*h*) $A \cap (B' \cup C')$

We know that,

$$A \cap (B' \cup C') = (A \cap B') \cup (A \cap C')$$

Using 4(*b*) of laws of set operations.

Consider the figures given below.



(*a*) (*b*) (*c*)

Fig. 8

$\therefore A \cap (B' \cup C')$ is the set of all persons who smoke cigarette and pipe but no cigar, cigarette and cigar but no pipe or cigarette alone.

(*i*) $A \cap (B \cap C')$:

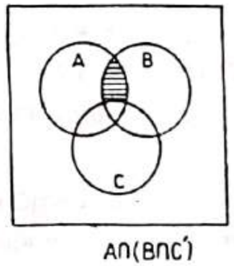
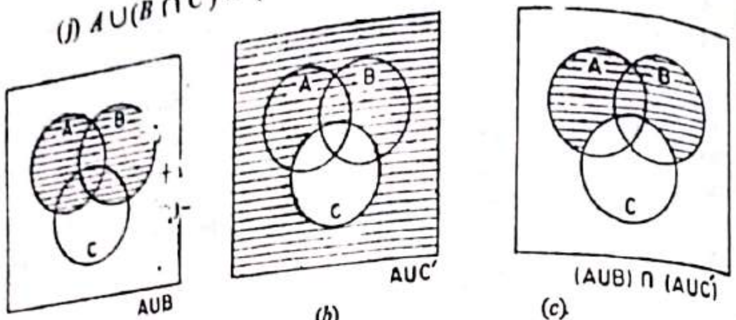


Fig. 9

The set of all persons who smoke cigarette and pipe but not cigar.

(j) $A \cup (B \cap C') = (A \cup B) \cap (A \cup C')$



(a) (b) (c)

Fig. 10

Thus $A \cup (B \cap C')$ represents the set of all persons who smoke cigarette or pipe, with no cigar.

Example 2. A company has interviewed 100 persons on their attitudes concerning a product they have introduced. Their attitudes are as indicated below.

	Liked	*Indifferent*	*Disliked*
Single males	10	5	5
Single Females	20	5	5
Married males	5	5	15
Married Females	10	5	10

Notation : S, set of single persons, M, males, F, females ;L, persons who liked ; I, persons, who were indifferent and D, persons who disliked.

(a) Find the number of people in each of the following sets

(i) S, M, F (ii) L, T, D

(iii) $S \cap L$, (iv) $S \cap F \cap D$

(v) $(S \cap M \cap D)'$ (vi) D'

(vii) $L \cup I$ (viii) $S \cap M \cap D'$

and (ix) $I \cup D$

(b) Verify (i) $n\{S \cap (L \cup I)\} = n\{(S \cap L) \cup (S \cap I)\}$

(ii) $n\{M \cup (F \cap L)\} = n\{(M \cup F)\{(M \cup L)\}$

Here we introduce a notation, which will be followed throughout. For any set A, $n(A)$ represents the number of elements in the set.

(i) $n(S) = 50$, $n(M) = 45$, $n(F) = 55$

(ii) $n(L) = 45$, $n(I) = 20$, $n(D) = 35$.

(iii) $S \cap L$ represents the set of persons who are single and liked the product $\therefore n(S \cap L) = 30$.

(iv) $S \cap F \cap D$ represents the set of persons who are single females who disliked $\therefore n(S \cap F \cap D) = 5$.

(v) $(S \cap M \cap D)'$ is the set of all persons who are not single males who disliked $\therefore n(S \cap M \cap D)' = 95$.

(vi) D' is the set of persons who do not dislike $\therefore n(D') = 65$.

(vii) $L \cup I$ is the set of all persons who either like or are indifferent *i.e.*, the set of all persons who do not dislike.

$\therefore \quad n(L \cup I) = 65 \quad$ by (vi)

(viii) $S \cap M \cap D'$ is the set of all single males who do not dislike $\therefore n(S \cap M \cap D') = 15$.

(ix) $I \cup D$ is the set of all persons who are either indifferent or who dislike.

$\therefore \quad n(I \cup D) = 55$.

(b) (i) $S \cap (L \cup I)$ is the set of all single persons who either like or are indifferent.

$\therefore \quad n(S \cap (L \cup I)) = 40$

$n(S \cap L) = 30 \quad$ from (iii) of (a)

$S \cap I$ is the set of all single persons who are indifferent

$\therefore \quad n(S \cap I) = 10$.

$(S \cap L) \cup (S \cap I)$ is the set of all single persons who like or the set of all single persons who are indifferent.

$\therefore n[S \cap L) \cup (S \cap I)] = 30 + 10 = 40 = n(S \cap (L \cup I))$

(ii) $n(F \cap L) = 30 \quad n(M) = 45 \; \therefore n(M \cup (F \cap L)) = 75$

$\therefore \quad n(M \cup F) = 100$

$M \cup L$ is the set of all persons who are males or the set of all persons who liked.

Note that there are 15 persons who are both males and who liked *i.e.*, $M \cap L \neq \phi$

$n(M \cap L) = 15$

$\therefore \quad n(M \cup L) = 90 - 15 = 75$.

Hence $\quad n(M \cup (F \cap L)) = 75 = n((M \cup F) \cap (M \cup L))$

Two useful results :

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

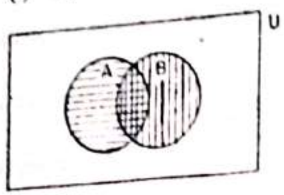


Fig. 11.

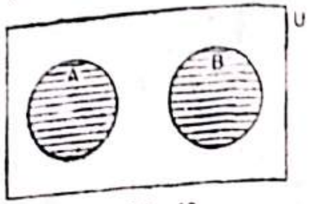


Fig. 12.

Refering to Fig. 11 for $A \cup B$, we see that when we add $n(A)$ and $n(B)$, $n(A \cap B)$ *i.e.*, the elements in $A \cap B$ had been counted twice, since $A \cap B$ is the subset of both A and B.

$$n(A \cup B) = n(A)+n(B)-n(A \cap B) \quad ...(1)$$

Suppose A and B are disjoint sets.

Then $A \cap B = \phi$ and $n(A \cap B) = 0$

and the above result becomes,

$$n(A \cup B) = n(A)+n(B)$$ Refer to Fig. 12

(2) $n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B)-n(A \cap C)$
$-n(B \cap C)+n(A \cap B \cap C)$.

This may be proved by the repeated application of result (1) given above.

$n[A \cup (B \cup C)] = n((A)+n(B \cup C)-n(A \cap (B \cup C))$ using (1)

$= n(A)+n(B)+n(C)-n(B \cap C)$
$-n((A \cap B) \cup (A \cap C))$
by result (1) and law 4(*b*) of set operations

$= n(A)+n(B)+n(C)-n(B \cap C)$
$-\{n(A \cap B)+n(A \cap C)-n((A \cap B) \cap (A \cap C)\}$ by result (1)

$= n(A)+n(B)+n(C)-n(B \cap C)-n(A \cap B)$
$-n(A \cap C)+n(A \cap B \cap C)$
$(\because A \cap A = A)$

Note. The truth of (2) can be verified by noticing the fact that we have "over subtracted" the elements common to all the sets A, B, and C if we consider

$$n(A)+n(B)+n(C)-n(A \cap B)-n(A \cap C)-n(B \cap C)$$
alone for $n(A \cup B \cup C)$.

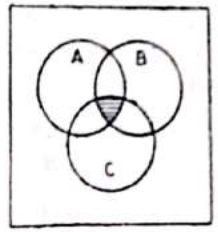


Fig. 13.

$A \cap B \cap C$ is the shaded portion

Example 3. A market research group conducted a survey of 1000 consumers and reported that 730 consumers liked product A and 455 consumers liked product B. What is the least number that must have liked both products assuming that there may be consumers of products different from A and B.

Let A represent the set of all consumers of the product A and B the set of all consumers of the product of B. $A \cap B$ gives the set of consumers who use both A and B.

We have,

$$n(A \cup B) = n(A)+n(B)-n(A \cap B)$$
$$\therefore \quad n(A \cap B) = n(A)+n(B)-n(A \cup B)$$
$$= 730+455-n(A \cup B)$$
$$= 1185-n(A \cup B).$$

When $n(A \cup B)$ is maximum, $n(A \cap B)$ will be minimum. If all the consumers liked A or B, $n(A \cup B)$ will be 1000 and this is the greatest value possible for $n(A \cup B)$.

$\therefore$ The least value of $n(A \cap B) = 1185-1000=185$

Thus 185 is the least number of people who liked both the products A and B.

Example 4. In a survey concerning the smoking habits of consumers it was found that, 55% smoke cigarette A, 50% smoke B. 42% smoke C, 28% smoke A and B, 20% smoke A and C, 12% smoke B and C and 10% smoke all the three cigarettes.

(i) What percentage do not smoke ?

(ii) What percentage smoke exactly two brands of cigarettes ?

Let A, B, C represent the set of all persons who smoke the brands A, B, C respectively. (*i*) $A \cup B \cup C$ is the set of all persons who smoke either A or B or C; or any two brands or all the brands. Therefore $(A \cup B \cup C)'$ will give the set of all people who do not smoke any one or any two or all the three brands *i.e.* the set of people who do not smoke.

Now, $n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B)$
$-n(A \cap C)-n(B \cap C)+n(A \cap B \cap C)$
$= 55+50+42-28-20-12+10$
$= 97$

$n(A \cup B \cup C)+n\{(A \cup B \cup C)'\} = 100$
$n\{(A \cup B \cup C)'\} = 100-n(A \cup B \cup C)$
$= 100-97$
$= 3$

3% do not smoke.

(*ii*) $A \cap B \cap C'$ is the set of persons who smoke A and B but not C.

$A \cap B \cap C$ is the set of persons who smoke all the three brands A, B, C.

The above sets are disjoint and their union is $A \cap B$. (Refer Fig. 14.)

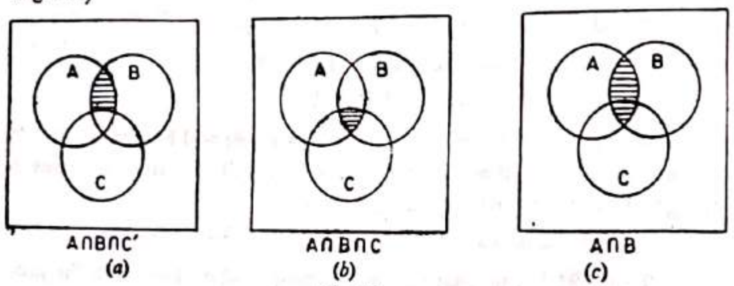


Fig. 14

$n(A \cap B \cap C')+n(A \cap B \cap C) = n(A \cap B)$
$n(A \cap B \cap C')+10 = 28$
$n(A \cap B \cap C') = 18$

Similarly $n(A \cap C \cap B') = 20-10 = 10$

and $n(B \cap C \cap A') = 12-10 = 2$

$\therefore$ The required number $= 18+10+2 = 30$

The same fact can be easily seen by drawing the following diagram.

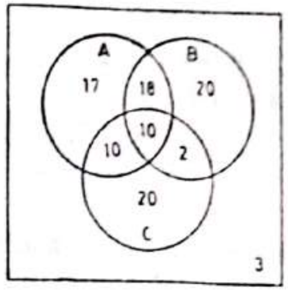


Fig. 15

Example 5. A company study of the product preferences of 10,000 consumers reported that each of the products A, B, C was liked by 5, 015 ; 3, 465 ; 4, 827 respectively and all the products were liked by 500 people, products A and B were liked by 1000, products A and C were liked by 850 and products B and C were liked by 1,420. Prove that the study results are not correct. It was found that an error was made in recording the number of consumers liking the products A and C. What is the value of this number ?

Let A, B, C denote the set of people who like products A, B, C respectively.

The given data means

$n(A) = 5015 \quad n(A \cap B) = 1000 \quad n(A \cap B \cap C) = 500$
$n(B) = 3465 \quad n(A \cap C) = 850 \quad n(A \cup B \cup C) = 10000$
$n(C) = 4827 \quad n(B \cap C) = 1420$

We also know that,

$n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B)-n(A \cap C)$
$-n(B \cap C)+n(A \cap B \cap C)$ ...(1)
$= 5,015+3,465+4,827-1000-850$
$-1,420+500$

i.e., $n(A \cup B \cup C) = 10,537 \neq 10,000$

This shows that there is an error.

According to the problem there is an error in recording the number $n(A \cap C)$. Now let us find the correct value of $n(A \cap C)$. Ignoring the given value of $n(A \cap C)$ and using the result (1)

$10,000 = 5,015+3,465+4,827-1,000-1420-n(A \cap C)+500$

$= 13,807 - 2420 - n(A \cap C)$

$n(A \cap C) = 11,387 - 10,000$

$= 1,387$

Problem Set 2 (c)

I. (*a*) (1) Find $A \cup B$ if

(*i*) $A = \{2, 4, 5\}$, $B = \{4, 7, 12, 13, 0\}$

(*ii*) $A = \{a, b, c\}$, $B = \{x, y, z\}$

(*iii*) $A = \{5, 6, 7, 8\}$, $B = \{6, 7\}$

(*iv*) A = {Jones, Mohammed, Krishnan}
B = {Krishnan, Paul, Sultan}

(*v*) $A = \{x : x \text{ is an even number}\}$
$B = \{y : y \text{ is an odd number}\}$

(*vi*) $A = \{x : x \text{ is a rational number}\}$
$B = \{y : y \text{ is an integer}\}$

(*vii*) $A = \{x : x \text{ is a consumer of a product } P\}$
$B = \{x : x \text{ is a consumer of a product } Q\}$

(2) (*a*) Find $A \cup B \cup C$ if

(*i*) $A = \{1, 2, 3\}$, $B = \{2, 3, -5, 0\}$
and $C = \{-5, \frac{1}{2}, \log 2, \Pi\}$

(*ii*) A is the set of positive integers
B is the set of negative integers and
C is the set of all rational numbers

(*b*) (1) Find $A \cap B$ for the sets A and B given in (*i*) through (*vii*) of problem I (*a*). (*i*)

(2) Find $A \cap B \cap C$ for the sets A, B and C given in (*i*) and (*ii*) of problem I (*a*) (2).

(*c*) (1) If the universal set $U = \{0, 1, 2, 3, 4, 5\}$ find the complement of A when

(*i*) $A = \{0, 1, 3\}$ (*ii*) $A = \{0, 1, 2, 3, 4\}$
(*iii*) $A = \phi$ (*iv*) $A = \{0, 1, 2, 3, 4, 5\}$

(2) What is the complement of the set of integers when the universal set is the set of all real numbers.

(3) What is the complement of the set of all male employees in a firm ? (Universal set is the set of all employees in the firm.)

(4) What is the complement of all skilled labourers in an industry taking the universal set as the set of all labourers in the industry.

(5) If $x = \{0, 1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$, $B = \{0, 5, 2, 4\}$. Verify De Morgan's Laws, by finding $A' \cup B'$, $A \cup B$, $A' \cap B'$ and $(A \cap B)'$.

(*d*) (1) Find $A - B$ and $B - A$ for the sets A and B given in the problem I (*a*) (1).

(2) Find $A \triangle B$ for the same set A and B given in I (*a*) (1).

II. (1) In a market survey people are to be classified whether they use the product of type I or type II or type III or none. Let A, B, C be the set of people who use the types I, II and III respectively. Describe the persons who belong to the following sets

(*a*) $A \cup B \cup C$ (*b*) $A' \cup B' \cup C'$ (*c*) AB
(*d*) AC (*e*) BC (*f*) ABC
(*g*) $A(B' \cup C')$ (*h*) $(AB') \cup C'$ (*i*) $A'B'C'$
(*j*) ABC' (*k*) $A \cup (BC')$ where AB denotes $A \cap B$

(2) A survey reports that 78% of those interviewed were married, 46% were married men, 12% were married women with no children and 30% were married women with children. Is the report correct ?

(3) In a market survey a manufactuer obtained the following data.

Did you use our brand ?	Percentage answering yes
1, April	59
2. May	62
3. June	62
4. April and May	35
5. May and June	33
6. April and June	31
7. April, May and June	22

Is this correct ?

(4) Draw the Venn diagram for the following data as in example 4. Assuming that every student in the class takes one of the courses find the total number of students in the class.

UNIT – 3

8

Matrices

8·1. Basic Concepts

Suppose a company owns three factories A, B, C which produce two products P and Q.

A produces 4 units of P and 5 units of Q;
B produces 7 units of P and 6 units of Q;
C produces 9 units of P and zero unit of Q.

The above information can be expressed in the form,

		Factories		
		A	B	C
Product	P	4	7	9
	Q	5	6	0

Each row represents the number of units of a particular product produced by the three different factories and each column represents the number of units of different products by a particular factory.

The rectangular arrangement of numbers,

$$\begin{bmatrix} 4 & 7 & 9 \\ 5 & 6 & 0 \end{bmatrix}$$

is called a matrix. The above matrix has 2 rows and 3 columns; and therefore it is called a 2×3 (read as 2 by 3) matrix. The first number 2 (in 2×3) indicates the number of rows and the second number 3 (in 2×3) indicates the number of columns. Let "a" represent the number of units of a product produced by one of the three factories.

Double suffix notation can be used to denote the type of the product and the factory that produces it. For example a_{23} denotes the number of units of the 2^{nd} product Q produced by the 3^{rd} factory C i.e. 0.

Similarly,

$$a_{11} = 4 \qquad a_{12} = 7 \qquad a_{13} = 9$$

$$a_{21} = 5 \quad \text{and} \quad a_{22} = 6.$$

In general a_{ij} denotes the number belonging to the i^{th} row and the j^{th} column in a matrix, and a_{ij} is called the address of an element. In the above case i can take the value 1, 2 and j the values 1, 2, 3. The matrix

$$\begin{bmatrix} 4 & 7 & 9 \\ 5 & 6 & 0 \end{bmatrix}$$

is a rectangular arrangement of six numbers 4, 7, 9, 5, 6, 0 in two rows and three columns. If the rectangular arrangement has m rows and n columns then it contains mn elements.

Definition. A rectangular arrangement of numbers in m rows and n columns is called a matrix of order $m \times n$. According to our notation any $m \times n$ matrix can be taken as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Hereafter we use capital letters $A, B, C, \ldots$ to denote matrices.

Example 1.

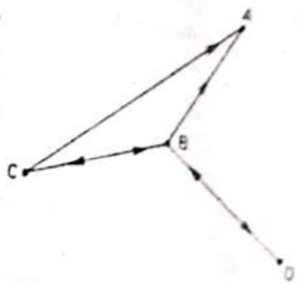
(a) $\begin{bmatrix} 4 & 3 & -2 \\ 2 & -3 & 11 \end{bmatrix}$ is a 2×3 matrix.

(b) $\begin{bmatrix} 9 & 4 \\ 6 & 2 \\ -5 & 0 \end{bmatrix}$ is a 3×2 matrix.

(c) $[7 \ \ 6 \ \ -5 \ \ 1]$ is an 1×4 matrix.

(d) $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ is a 2×1 matrix.

Example 2. In the following diagram the lines connecting the points A, B, C, D are considered as lines of communication and the arrow head represents the direction of communication.



From B one can speak to A; but not in the reverse direction. Taking B and C, it is a two-way communication. Now we translate the above problem into the matrix form.

$a_{ij} = 1$ if the communication is from the i^{th} station to the j^{th} station.

$a_{ij} = 0$ if the communication from the i^{th} station to the j^{th} station is not possible.

We assume that there is no communication from a station to itself i.e. $a_{ij} = 0$ if $i = j$.

Then we get,

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

as the matrix representation of the above communication problem.

Now we give certain terms associated with a matrix. The entries in a matrix will be called **elements** of the matrix. (Sometimes they are called scalars.) If all the elements of a matrix are zero the matrix is called a **null matrix** (or zero matrix) and is denoted by **0**. Bold letter **0** is used just to distinguish the null matrix from the zero element. If the number of rows and columns are equal, *i.e.* if $m = n$, we say that the martix is a **square martix** of order n. When all the elements other than those in the main diagonal* of a square martix are zero the martix is called a **diagonal martix**. A square matrix is said to be triangular if all the entries either above or below the main diagonal are zero. A **scalar matrix** is a diagonal matrix in which all the diagonal elements are equal. If in the above diagonal matrix all the diagonal elements are unity (one) we say the matrix is an **identity matrix or unit matrix**. The matrix obtained from a given matrix A by interchanging its rows and columns is called the **transpose** of A and is denoted by A'.

Let $A = (a_{ij})$ $\begin{array}{l} i = 1, n \\ j = 1, n \end{array}$ be a square matrix.

If $a_{ij} = a_{ji}$ the marix is said to be **symmetric**.

If $a_{ij} = -a_{ji}$ in the matrix A it is said to be **skew-symmetric**.

Examples.

(*a*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[0\ 0\ 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

are null matrices.

(*b*) $\begin{bmatrix} -3 & 2 & -5 \\ 4 & 1 & 6 \\ 7 & 8 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

are square matrices of order 3 and 2 respectively.

(*c*) $\begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

*Main diagonal of a square matrix is the diagonal which is composed of all the element a_{ij} of the matrix for which $i=j$.

and $\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

are diagonal matrices of order 2, 3 and 4 respectively.

(*d*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are identity matrices denoted by I_2, I_3 and I_4.

(*e*) If $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 5 & 6 \end{bmatrix}$, its transpose

$$A' = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

(*f*) $\begin{bmatrix} 2 & 1 & 6 \\ 1 & 3 & -8 \\ 6 & -8 & 2 \end{bmatrix}$ $\begin{bmatrix} 0 & -5 \\ -5 & 2 \end{bmatrix}$

and $\begin{bmatrix} 1 & 6 & -7 & 5 \\ 6 & 2 & 0 & 9 \\ -7 & 0 & -3 & -10 \\ 5 & 9 & -10 & 4 \end{bmatrix}$

are symmetric matrices.

(g) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 5 & -6 & 9 \\ -5 & 0 & 7 & 2 \\ 6 & -7 & 0 & -3 \\ -9 & -2 & 3 & 0 \end{bmatrix}$

are skew symmetric matrices.

Let A be a given matrix. By deleting a few rows and columns we get a new matrix called the **sub-matrix** of A. Clearly A is also a sub-matrix of itself.

Consider $A = \begin{bmatrix} 3 & 5 & 6 \\ 4 & 7 & 8 \end{bmatrix}$

The sub-matrices are,

$\begin{bmatrix} 3 & 5 & 6 \\ 4 & 7 & 8 \end{bmatrix}$ [3 5 6], [4 7 8]

[3 5], [3 6], [5, 6], [4, 7], [4,8], [7, 8]

$\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

[3], [5], [6], [4], [7], [8]

Definition. Two matrices of the same order are said to be **equal** only when the corresponding elements (elements with the same address) are equal.

Let $A = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ -4 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 4 & 7 \end{bmatrix}$

Here $A = C$, but $A \neq B$ and $B \neq C$.

Consider the matrix [5 −7 17 0 9]. This is a 1×5 matrix having one row only. Such matrices consisting of one row only are

called row matrices. $[a_1\ a_2\ \ldots\ldots\ a_n]$ is a row matrix with n elements.

$\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$ is a 3×1 matrix, having one column only.

Such matrices having only one column are called column matrices.

$\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$ is a column, matrix with n elements.

matrix with n elements.

Problem Set 8 (*a*)

(1) Find a_{12}, a_{23}, a_{51}, a_{53} in the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 9 & 10 \\ 7 & -11 & 12 & 8 & 0 & -10 \\ 3 & 5 & 7 & 0 & -9 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 100 & 10 & \frac{1}{2} & 6 & 11 & 15 \end{bmatrix}$$

(2) Find the type of the matrices given below :

(*i*) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 5 & 6 & 7 \\ 6 & 0 & 12 \\ 7 & 12 & -1 \end{bmatrix}$

(*iii*) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ (*iv*) $\begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & -12 \\ -7 & 12 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(3) Find all submatrices of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

How many of these are square submatrices ?

8·2. Addition of Matrices

Two matrices can be added (subtracted) only if they are of the same order. The sum (difference) of two $m \times n$ matrices is another $m \times n$ matrix whose elements are the sum (difference) of the corresponding elements of the matrices.

Example 1.

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 5 & -6 \\ 2 & -7 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 1 \\ 5 & -2 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

A and B are 3×3 matrices. Therefore $A+B$ and $A-B$ are also 3×3 matrices.

$$A+B = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 5 & -6 \\ 2 & -7 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 5 & -2 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+(-1) & -1+0 & 0+1 \\ -3+5 & 5+(-2) & -6+2 \\ 2+3 & -7+4 & 8+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 3 & -4 \\ 5 & -3 & 11 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 5 & -6 \\ 2 & -7 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 5 & -2 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-(-1) & -1-0 & 0-1 \\ -3-5 & 5-(-2) & -6-2 \\ 2-3 & -7-4 & 8-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ -8 & 7 & -8 \\ -1 & -11 & 5 \end{bmatrix}$$

Example 2. Suppose the regional sales manager of a Drug company wishes to get a day-by-day inventory from his dealers, of different type of medicines in different packings. He can get the report from each dealer in the following form.

50 tablet bottles	*100 tablet bottles*	*500 tablet bottles*	
a_{11}	a_{12}	a_{13}	—Vitamin C pills
a_{21}	a_{22}	a_{23}	—Pill X—for digestive disorder
a_{31}	a_{32}	a_{33}	—Pill V—for sore throat
a_{41}	a_{42}	a_{43}	—Pill Z—for liver complaint

If the report is from the k^{th} dealer it can be denoted by A_k. If there are 150 dealers in the region, the total inventory matrix T is given by,

$$T = A_1 + A_2 + \ldots\ldots + A_k + \ldots\ldots + A_{150}$$

Example 3. An engineering company at Madras gets 3 types of components from Bombay and Calcutta, details of which are given in the following matrix form.

	Purchase cost	*Transportation cost*	
From Bombay $A =$	22	15	Component I
	23	13	Component II
	18	12	Component III
From Calcutta $B =$	21	20	Component I
	22	17	Component II
	16	11	Component III

The total purchase and transportation cost is,

$$A+B=\begin{bmatrix} 43 & 35 \\ 45 & 30 \\ 34 & 23 \end{bmatrix}$$

Some Properties of Addition

Let $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$ $C=\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$

Then,

$$A+B=\begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix} \quad B+A=\begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix}$$

i.e., $A+B=B+A$. This is true for any two matrices which can be added.

Hence Addition is commutative.

$$(A+B)+C=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}+\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix}+\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 7 & 3 \\ -1 & 7 \end{bmatrix}$$

$$A+(B+C)=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}+\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 7 & 3 \\ -1 & 7 \end{bmatrix}$$

i.e. $(A+B)+C=A+(B+C)$.

This is true for any three matrices which can be added. Hence addition is associative.

Problem Set 8 (b)

1. If $A=\begin{bmatrix} 2 & 3 & -4 \\ 6 & 7 & 8 \end{bmatrix}$, $B=\begin{bmatrix} 6 & -3 & 2 \\ 5 & 0 & 8 \end{bmatrix}$

$$C=\begin{bmatrix} 1 & 2 & -3 \\ 5 & -4 & 3 \end{bmatrix}$$

find (*a*) $A+B-C$. (*b*) $A-B+C$. (*c*) $B-C+A$.

(*d*) $A-B-C$. (*e*) verify commutative property.

(*f*) verify associative property.

(2) If $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

Show that (*i*) $(A+B)'=A'+B'$

(*ii*) $(A-B)'=A'-B'$

8·3. Scalar Multiplication

Let $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

$$A+A+A+A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 4\times2 & 4\times3 \\ 4\times-1 & 4\times4 \end{bmatrix}=\begin{bmatrix} 8 & 12 \\ -4 & 16 \end{bmatrix}$$

Can we denote $A+A+A+A$ as $4A$ to mean that the matrix A had been multiplied by 4 ? Notice that 4 is a number and A is an arrangement (matrix) and we are multiplying two things of different species. This interpretation is justified if we define scalar multiplication as follows.

A matrix is said to be multiplied by a scalar if every element (or entry) in the matrix is multiplied by the same scalar.

$$\text{Thus } 4A=4\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}=\begin{bmatrix} 2\times4 & 3\times4 \\ -1\times4 & 4\times4 \end{bmatrix}=\begin{bmatrix} 8 & 12 \\ -4 & 16 \end{bmatrix}$$

and therefore $A+A+A+A$ can be written as $4A$. We demonstrated the multiplication by a scalar when the scalar was an integer. The rule holds even when the scalar is any number.

Problem Set 8 (c)

Let $A = \begin{bmatrix} 2 & 0 & 9 \\ -1 & 6 & 11 \\ 4 & 8 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 0 \\ 1 & -1 & -4 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & 0 & -5 \\ 3 & 7 & 2 \\ -1 & 0 & -1 \end{bmatrix}$

(1) Find (*a*) $3A$. (*b*) $4A$. (*c*) $\frac{1}{2}C$. (*d*) $A+2B$. (*e*) $2B-3C$. (*f*) $A+3B-2C$. (*g*) $2A-B+5C$.

(2) Verify $(kA)' = kA'$ where k is a scalar and $A = \begin{bmatrix} 2 & 5 & 3 \\ 6 & 7 & 8 \end{bmatrix}$

Multiplication of a matrix by a matrix Let the prices of 3 fruits at four cities be given by the matrix.

Cost per box in rupees.

	Apple	Orange	Mango	
$P =$	20	12	8	Madras.
	17	10	9	Bangalore.
	16	11	7	Hyderabad.
	20	12	10	Cochin.

Let us call this as price matrix.

The supply position of these fruits with two dealers A and B is given by the matrix,

Dealer

	A	B	In number of boxes.
$Q =$	100	200	Apples.
	200	200	Oranges.
	300	100	Mangoes.

Q is called the quantity matrix.

Let us have a closer look at each matrix. In matrix P, each row gives the price of the three kinds of fruits in a particular

city and each column gives the price of a particular fruit in the four cities. Likewise, in Q each row gives the quantity available in each kind of fruit with both the suppliers and each column gives the quantity a particular dealer can supply in the various kinds of fruits.

Consider the two matrices,

$$[20 \quad 12 \quad 8] \text{ and } \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

which give the prices at Madras and the quantity available with dealer A respectively. If we want to compute the income of the dealer A from Madras we multiply the price of each box of fruit by the quantity sold in the three kinds of fruit and add them up.

Thus the income $= 20\times100+12\times200+8\times300$

$= 6800.$

So we write,

$$[20 \quad 12 \quad 8] \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = [6800]$$

and we have multiplied a 1×3 matrix by another 3×1 matrix and the result is a 1×1 matrix.

Let us go one step further. Consider the matrices,

$$[20 \quad 12 \quad 8] \text{ and } \begin{bmatrix} 100 & 200 \\ 200 & 200 \\ 300 & 100 \end{bmatrix}$$

These matrices give the prices at Madras and the quantity that can be supplied by A and B. Performing the operation we have indicated earlier we get the income of A and B from Madras.

So we write,

$$[20 \quad 12 \quad 8] \times \overset{\quad A \qquad B}{\begin{bmatrix} 100 & 200 \\ 200 & 200 \\ 300 & 100 \end{bmatrix}} = \overset{\; A \qquad B}{[6800 \quad 7200]}$$

By multiplying a 1×3 matrix and a 3×2 matrix we get a 1×2 matrix.

Now we take two cities and one dealer ; the prices at Madras and Bangalore are given by

$$\begin{bmatrix} 20 & 12 & 8 \\ 17 & 10 & 9 \end{bmatrix}$$

and the quantity available with A is given by

$$\begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

Working as before we get,

$$\begin{bmatrix} 20 & 12 & 8 \\ 17 & 10 & 9 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 6800 \\ 6400 \end{bmatrix}$$

The entries in the product give the income of A from Madras and Bangalore. The product of a 2×3 matrix and a 3×1 matrix is a 2×1 matrix.

Now considering two cities and two dealers the price matrix will be a 2×3 matrix, the quantity matrix will be a 3×2 matrix and the product will be a 2×2 matrix.

We have,

$$\begin{bmatrix} 20 & 12 & 8 \\ 17 & 10 & 9 \end{bmatrix} \times \begin{bmatrix} 100 & 200 \\ 200 & 200 \\ 300 & 100 \end{bmatrix} = \begin{bmatrix} 6800 & 7200 \\ 6400 & 6300 \end{bmatrix}$$

Taking the situation as a whole we get

$$\begin{array}{l} \\ \text{Mad.} \\ \text{Bang.} \\ \text{Hyd.} \\ \text{Coch.} \end{array} \begin{array}{c} \begin{array}{ccc} App & Org & Mang \end{array} \\ \begin{bmatrix} 20 & 12 & 8 \\ 17 & 10 & 9 \\ 16 & 11 & 7 \\ 20 & 12 & 10 \end{bmatrix} \\ 4\times3 \end{array} \times \begin{array}{l} \\ \text{App.} \\ \text{Org.} \\ \text{Mang.} \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 100 & 200 \\ 200 & 200 \\ 300 & 100 \end{bmatrix} \\ 3\times2 \end{array}$$

$$= \begin{array}{l} \\ \text{Mad} \\ \text{Bang} \\ \text{Hyd} \\ \text{Coch} \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 6800 & 7200 \\ 6400 & 6300 \\ 5900 & 6100 \\ 7400 & 7400 \end{bmatrix} \\ 4\times2 \end{array}$$

i.e, $P \times Q = R$

"Price matrix" "Quantity matrix" "Income matrix" (or Revenue matrix)

The price matrix P is of order 4×3, the quantity matrix Q is of order 3×2 and their product R *viz.,* the income matrix is of order 4×2. Only when the columns in the matrix P on the left and the number of rows in the matrix Q on the right are equal, the two matrices can be multiplied in the way indicated, *i.e.,* PQ is defined.

Now let us take any entry (element) from the income matrix, for example, 6100. It is in the third row and second column, and it gives the income from Hyderabad for the dealer B. This is got by multiplying by the third row of the price matrix (left matrix) and the second column of the quantity matrix (right matrix). In short the entry in the third row and second column of the product matrix arises out of the multiplication of the third row of the left matrix and the second column of the right matrix, which we diagrammatically represent,

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \boxed{\times \;\; \times \;\; \times} \\ \times & \times & \times \end{bmatrix} \times \begin{bmatrix} \odot & \boxed{\odot} \\ \odot & \boxed{\odot} \\ \odot & \boxed{\odot} \end{bmatrix} = \begin{bmatrix} \oplus & \oplus \\ \oplus & \oplus \\ \oplus & \boxed{\oplus} \\ \oplus & \oplus \end{bmatrix}$$

4×3 Third row $\quad$ 3×2 Second column $\quad$ 4×2 Element in the 3rd row and 2nd column.

Now we are in a position to define matrix multiplication.

Definition. Let A be an $m \times p$ matrix and B be a $p \times n$ matrix. The product $C = AB$ is an $m \times n$ matrix where the entry C_{ij} of C is obtained by multiplying the corresponding entries of the i^{th} row of A by those of j^{th} column of B and then adding the results.

$$\underset{m \times p}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \boxed{a_{i1} \;\; a_{i2} \;\; \cdots \;\; a_{ip}} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}} \times \underset{p \times n}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix}}$$

$$= \underset{m \times n}{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1j} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2j} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{i1} & C_{i2} & \cdots & \boxed{C_{ij}} & \cdots & C_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{m1} & C_{m2} & \cdots & C_{mj} & \cdots & C_{mn} \end{bmatrix}}$$

where $C_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots\ldots + a_{ip} b_{pj}$

Facts. If A, B, C are three matrices, then

$A(BC) = (AB)\ C$, provided the multiplications indicated are defined.

Example 1. If

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

Find AB. A is a 2×2 matrix and B is a 2×3 matrix. Therefore AB is a 2×3 matrix.

$$AB = \begin{bmatrix} 2-6 & 0+9 & 4+3 \\ -1-2 & 0+3 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 7 \\ -3 & 3 & -1 \end{bmatrix}$$

Example 2. If

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 6 & 2 & 8 \\ 0 & 2 & -1 \\ 3 & -4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 & 0 & 8 & -5 \\ -1 & 6 & 2 & 1 & 6 \\ 2 & -7 & -1 & 4 & -7 \end{bmatrix}$$

find AB.

A is a 4×3 matrix, B is a 3×5 matrix and hence AB is a 4×5 matrix.

$$AB = \begin{bmatrix} 8+1+8 & 2-6-28 & 0-2-4 & 16-1+16 & -10-6-28 \\ 24-2+16 & 6+12-56 & 0+4-8 & 48+2+32 & -30+12-56 \\ 0-2-2 & 0+12+7 & 0+4+1 & 0+2-4 & 0+12+7 \\ 12+4+10 & 3-24-35 & 0-8-5 & 24-4+20 & -15-24-35 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -32 & -6 & 31 & -44 \\ 38 & -38 & -4 & 82 & -74 \\ -4 & 19 & 5 & -2 & 19 \\ 26 & -56 & -13 & 40 & -74 \end{bmatrix}$$

Can you find BA in this case ? No. So AB may be defined but BA may not be defined. In some problems AB and BA both may be defined. But still AB need not be equal to BA as illustrated below.

An Important Note. In the product AB, A is said to have been post-multiplied by B, and B is said to have been premultiplied by A.

i.e., AB is called the post-multiplication of A by B or premultiplication of B by A.

Examples :

1. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -3 \end{bmatrix}$

A is a 3×2 matrix and B is a 2×3 matrix. The product AB is a 3×3 matrix while BA is a 2×2 matrix. Thus, even though the products AB and BA are defined they are not equal.

2. Let $A = \begin{bmatrix} -3 & 7 \\ 4 & -5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $AI = \begin{bmatrix} -3+0 & 0+7 \\ 4+0 & 0-5 \end{bmatrix}$ $\begin{matrix} +7 \\ -5 \end{matrix}\Big]$

$$= \begin{bmatrix} -3 & 7 \\ 4 & -5 \end{bmatrix} = A$$

when a martix A is multiplied by the identity martix I, the product is A itself. In other words, multiplication of a matrix by a unit matrix (if the multiplication is defined) leaves the matrix unaltered.

3. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} +1 & -\frac{2}{3} \\ -1 & 1 \end{bmatrix}$

$$CA = \begin{bmatrix} 3-2 & 2-2 \\ 3-3 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The product of A and C is the identity matrix.

4. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} -1+1 & 2-2 \\ 1-1 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

The matrics A and B are non-null matrices ; but their product AB is a null matrix *i.e.*, $AB = O$ does not always imply that either $A = O$ or $B = O$.

5. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -5 & 2 & 3 \end{bmatrix}$,

$$B = \begin{bmatrix} 2 & 5 & -1 & -7 \\ -2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

and $C = \begin{bmatrix} 3 & 6 & 0 & -6 \\ -1 & 2 & 4 & 5 \\ 4 & 3 & 2 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2+4+3 & 5-2+2 & -1-6+1 & -7-8+2 \\ 4-2-9 & 10+1-6 & -2+3-3 & -14+4-6 \\ -10-4+9 & -25+2+6 & 5+6+3 & 35+8+6 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 & -6 & -13 \\ -7 & 5 & -2 & -16 \\ -5 & -17 & 14 & 49 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3+2+4 & 6-4+3 & 0-8+2 & -6-10+3 \\ 6-1-12 & 12+2-9 & 0+4-6 & -12+5+9 \\ -15-2+12 & -30+4+9 & 0+8+6 & 30+10+9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 & -6 & -13 \\ -7 & 5 & -2 & -16 \\ -5 & -17 & 14 & 49 \end{bmatrix}$$

$\therefore$ $AB = AC$ But $B \neq C$

i.e., cancellation of A is not valid in general.

2. (*i*) $AX = B$ is a matrix equation where

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 7 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Write down the two linear equations separately.

(*ii*) $AX = B$ is a matrix equation where

$$A = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 0 & 4 & 5 & 1 \\ 1 & 0 & 0 & 7 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ -9 \\ 13 \end{bmatrix}$$

Write down the three linear equations separately.

(*iii*) If $PY = R$ where $P = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & 5 \\ 0 & -3 & 4 \\ 5 & 2 & 0 \end{bmatrix}$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad R = \begin{bmatrix} 7 \\ 11 \\ -9 \\ 5 \end{bmatrix}$$

Write down the four linear equations separately.

8·5. Inverse of a Matrix

Now let us consider a situation where matrices can be usefully employed in solving a system of linear equations.

Suppose, the management of a company faces the following problem. One of its factories uses two different machines *M* and *N* to produce two different items A and B. Machines M and N can operate 14 and 18 hours respectively everyday. The machine M requires 3 hours to produce one unit of A and 2 hours to produce one unit of B. The machine N requires 3 hours to produce one unit of A and 3 hours to produce one unit of B. Then how many units of each product should be produced so that the machines work to the capacity.

We proceed as follows. Let x be the number of units of product A and y the number of units of product B that are produced. Taking the machine M it requires $3x$ hours to produce x units of A and $2y$ hours to produce y units of B. Then,

$$3x+2y = 14$$

Likewise considering the machine N, we get

$$3x+3y = 18$$

Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 14 \\ 18 \end{bmatrix}$

Then the above two equations can be put as a single matrix equations,

$$AX = B$$

Let us pre-multiply both sides by a matrix C (to be found out).

Then, $$CAX = CB \quad \ldots(1)$$

If $CA = I$, the identity matrix, we will have $CAX = IX = X$.

Then the equation (1) becomes,

$$X = CB \quad \ldots(2)$$

From the example No. 3, given in the previous section 10.4, we see that,

if $$A = \begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix}$$

then C should be

$$\begin{bmatrix} 1 & \frac{-2}{3} \\ -1 & 1 \end{bmatrix}$$

so that $CA = I$

From (2), $X = CB$ gives,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{-2}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -12 \\ -14 & +18 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\therefore$ $$x = 2 \text{ and } y = 4$$

So 2 units of product A and 4 units of product B should be produced to keep the machines M and N to work to their capacity.

In the above discussion we have assumed the existence of a matrix C such that $CA = I$.

Definition. A matrix C is said to be the inverse of A if $CA = AC = I$ and C is denoted by A^{-1}.

From the definition it can be easily shown that (1) A has to be a square matrix and we can speak of the inverse of a matrix only if the matrix is a square matrix. (2) A may or may not have an inverse. (3) If A has an inverse it is unique. (4) The inverse of the inverse of a matrix A is A itself *i.e.* $(A^{-1})^{-1} = A$.

Given a matrix A we cannot go on guessing a matrix A^{-1} so that $A^{-1}A = AA^{-1} = I$. We should evolve some technique of finding the inverse. Of the various techniques available we give here only two.

Method I. *Finding inverse using elementary row (column) operations.*

The elementary row operations are defined below :

(*i*) Interchange of two rows (denoted by R_{ij} which means the i^{th} and the j^{th} rows are interchanged).

(*ii*) Multiplication of a row by a non-zero scalar. (denoted by $k R_i$ which means that i^{th} row is multiplied by k)

(*iii*) Replacement of the i^{th} row by the sum of the i^{th} row and k times the j^{th} row. This operation is denoted by $R_i+(k) R_j$. For example, $R_2+(5) R_3$ means that, to the second row we add 5 times the third row. Elementary column operations are defined likewise.

Steps in the Computation of Inverse

1. Divide the first row of the matrix by the entry in its first column if necessary and if it is not zero. If it is zero interchange the rows to get a non zero element in the first row first column to get 1 as the element in the first row first column. Use the new first row and obtain zeros in the first column of the other rows.

2. Divide the second row by the entry in its second column and use the resulting row to obtain zeros in the second column of each of its other rows and so on.

3. Perform the same operations in the same order on a suitable unit matrix I (*i.e.* the unit matrix which can multiply A) *i.e.* Reduce $(A : I)$ to $(I : B)$ using elementary row operations only. Then $B = A^{-1}$ If the method fails then A^{-1} does not exist.

The method is illustrated by the following examples.

Example 1. Find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 2 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc:cc} 1 & 1 & \frac{1}{2} & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \left(\frac{1}{2}\right) R_1$$

$$\left[\begin{array}{cc:cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 2 & \frac{-3}{2} & 1 \end{array}\right] R_2+(-3) R_1$$

$$\left[\begin{array}{cc:cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{-3}{4} & \frac{1}{2} \end{array}\right] \frac{1}{2} R_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & \frac{5}{4} & \frac{1}{-2} \\ 0 & 1 & \frac{3}{-4} & \frac{1}{2} \end{array}\right] R_1+(-1) R_2$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

Example 2. Find the inverse of $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 4 & 0 & 2 & 1 & 0 & 0 \\ 2 & 10 & 2 & 0 & 1 & 0 \\ 3 & 9 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 1 \\ 2 & 10 & 2 & 0 & 1 & 0 \\ 3 & 9 & 1 & 0 & 0 & 1 \end{array}\right] \frac{1}{4} R_1$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 10 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 9 & -\frac{1}{2} & -\frac{3}{4} & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2+(-2) R_1 \\ R_3+(-3) R_1 \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{10} & \frac{-1}{20} & \frac{1}{10} & 0 \\ 0 & 9 & \frac{1}{-2} & -\frac{3}{4} & 0 & 1 \end{array}\right] \frac{1}{10} R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{4} & 1 & 0 \\ 0 & 1 & \frac{1}{10} & \frac{-1}{20} & \frac{1}{10} & 0 \\ 0 & 0 & \frac{-14}{10} & \frac{-3}{10} & \frac{-9}{10} & 1 \end{array}\right] R_3+(-9)\,R_1$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{10} & \frac{-1}{20} & \frac{1}{10} & 0 \\ 0 & 0 & 1 & \frac{3}{14} & \frac{9}{14} & \frac{-5}{7} \end{array}\right] \frac{-10}{14}R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{7} & \frac{-9}{28} & \frac{5}{14} \\ 0 & 1 & 0 & \frac{-1}{14} & \frac{1}{28} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{3}{14} & \frac{9}{14} & \frac{-5}{7} \end{array}\right] \begin{array}{l} R_1+\left(-\frac{1}{2}\right)R_3 \\ R_2+\left(\frac{-1}{10}\right)R_3 \\ \end{array}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{-9}{28} & \frac{5}{14} \\ \frac{-1}{14} & \frac{1}{28} & \frac{1}{14} \\ \frac{3}{14} & \frac{9}{14} & \frac{-5}{7} \end{bmatrix}$$

Problem Set 8' (f)

Compute the inverses of the following matrices if they exist.

(1) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

(15) $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix}$ (B.B.A, Sept. 1971 and April 1972)

(16) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ (B.B.A. April 1972)

Method II

Finding the Inverse by Adjoint Matrix Method

As a preliminary we define what is meant by the determinant of a matrix. It is defined only for square matrices. Just take the matrices as they appear and consider the arrangement as a determinant; and evaluate the determinant. Thus with every square matrix we can associate a unique number. If A is a square matrix, its determinant is denoted by $|A|$.

For example,

if $$A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10-12 = -2$$

Let A be a square matrix of order n; and Mij denote the sub-matrix corresponding to the element a_{ij} *i.e.* the matrix got by suppressing the row and column through a_{ij}. The number $(-1)^{i+j}|M_{ij}|$ is called the cofactor of a_{ij} and is written as C_{ij}. The matrix formed by C_{ij}'s got by replacing a_{ij} by C_{ij} is called the cofactor matrix of A and is written as *Cof A*. The transpose of cof A is called adjoint of A and is written as **adj. A.** We have the theorem (proof omitted):

For the square matrix A, A^{-1} is given by

$$\frac{1}{|A|} \text{ adj } A \text{ provided } |A| \neq 0.$$

Thus $$A^{-1} = \frac{1}{|A|} \text{ adj. } A.$$

An important fact.

A^{-1} exists if and only if $|A| \neq 0$.

i.e. If A^{-1} exists than $|A| \neq 0$ and if $|A| \neq 0$, then A^{-1} exists.

Example 1. Find the inverse of

$$\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = 2\times5-2\times3 = 10-6 = 4$$

$$C_{11} = (-1)^{1+1}\, 5 = 5$$
$$C_{21} = (-1)^{2+1}\, 2 = -2$$
$$C_{12} = (-1)^{1+2}\, 3 = -3$$
$$C_{22} = (-1)^{2+2}\, 2 = 2$$

$$\therefore \text{ Cot } A = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix}$$

$$\text{adj } A = (\text{Cof } A') = \begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$= \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-1}{2} \\ \frac{-3}{4} & \frac{1}{2} \end{bmatrix}$$

Example 2. Find the inverse of

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix}$$

$$= 4\,(10-18) - 0 + 2\,(18-30)$$

$$= -32-24 \quad = -56$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} = 10-18 = -8$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2-6) = 4$$

$$C_{13} = (-)^{1+3}\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix} = 18-30 = -12$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ 9 & 1 \end{vmatrix} = -(0-18) = 18$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4-6 = -2$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 0 \\ 3 & 9 \end{vmatrix} = -(36-0) = -36$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 10 & 2 \end{vmatrix} = (0-20) = -20$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = -(8-4) = -4$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 0 \\ 2 & 10 \end{vmatrix} = (40-0) = 40$$

$$\therefore \quad \text{Cof } A = \begin{bmatrix} -8 & 4 & -12 \\ 18 & -2 & -36 \\ -20 & -4 & 40 \end{bmatrix}$$

$$\text{adj } A = (\text{cof } A)' = \begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A$$

$$= \frac{1}{-56}\begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7} & \frac{-9}{28} & \frac{5}{14} \\ \frac{-1}{14} & \frac{1}{28} & \frac{1}{14} \\ \frac{3}{14} & \frac{9}{14} & \frac{-5}{7} \end{bmatrix}$$

8·6. Solution of a System of Linear Equations I (By Matrix inversion Technique)

Let $\qquad AX = B \qquad$ ...(1)

be the given equation. Let A^{-1} be the inverse of A.

Premultiplying both sides of (1) by A^{-1}.

$$A^{-1}AX = A^{-1}B$$

$$(A^{-1}A)\,X = A^{-1}B$$

$$IX = A^{-1}B.$$

$$X = A^{-1}B. \qquad \text{...(2)}$$

Example 1. Solve

$$2x_1+3x_2-x_3 = 9$$

$$x_1-x_2+x_3 = 9$$

$$3x_1-x_2-x_3 = -1 \qquad \textit{(B.B.A. April 1970)}$$

The above system in the matrix notation is, $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

We know that $X = A^{-1}B$ [refer (2)]

Therefore finding the solution to the given system amounts to (*i*) Finding A^{-1} and (*ii*) post multiplying A^{-1} by B.

Let us find the inverse of A.

$$C_{11}=(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}=0 \qquad C_{21}=(-1)^{2+1}\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}=+4$$

$$C_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}=4 \qquad C_{22}=(-1)^{2+2}\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix}=1$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}=-4 \qquad C_{23}=(-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}=11$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix} 1 & -1 \\ 3 & +1 \end{vmatrix}=-4 \qquad C_{32}=(-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}=-3$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}=-1$$

$$\text{Cof } A=\begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix} \quad \therefore \text{adj } A=\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$|A|=2(-1+1)-3(-1-3)-1(-1-3)=0+12+4=16$$

$$\therefore A^{-1}=\frac{1}{|A|}\text{ adj } A=\frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$A^{-1}B=\frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$=\frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

But $X=A^{-1}B$.

i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore$ $x_1=2$, $x_2=3$ and $x_3=4$.

3

Differential Calculus

3·1 Idea of Limit

As the production quantities increase and approach plant capacity the manufacturing cost function may suffer changes in form. A linear cost model useful for small quantities may become quadratic as the items manufactured approach plant capacity. Here we describe the theory of limits in order to construct models for such situations.

Consider the sequence of numbers $1, \frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots\ldots$

This sequence has a pattern, $\frac{1}{10^n}$ where n is zero or a positive integer. By giving values to n from 0, we get different terms of the sequence. Such a term from which we can get different terms of the sequence by giving suitable values to n is called the general term of the sequence.

1 is the first term of the sequence.

$\frac{1}{10^1}$ **is the second term of the sequence.**

$\frac{1}{10^2}$ **is the third term of the sequence.**

... ...

$\frac{1}{10^n}$ **is the $(n+1)^{th}$ term of the sequence.**

We can go endlessly, by following this rule, to write the terms of this sequence, that is, this sequence has no terminating term. Such a sequence is called an infinite sequence. As n increases

indefinitely the terms of the sequence become smaller and smaller and they move closer and closer to zero.

This phenomenon of n becoming indefinitely larger and larger is termed as, "n tends to infinity" and it is symbolically written as "$n \to \infty$." Zero is called the limit of the sequence $1, \frac{1}{10}, \frac{1}{10^2}, \ldots \frac{1}{10^n}, \ldots,$ for the difference between zero and the term $\frac{1}{10^n}$ can be made as small as we please by taking n sufficiently large. In short, we write,

$$\text{as } n \to \infty, \frac{1}{10^n} \to 0 \text{ or } \qquad \underset{n \to \infty}{\text{Limit}} \frac{1}{10^n} = 0$$

Again consider the sequence,

$$1, \frac{1}{0 \cdot 1}, \frac{1}{0 \cdot 01}, \frac{1}{0 \cdot 001}, \frac{1}{0 \cdot 0001}, \ldots\ldots$$

i.e. $$1, \frac{1}{\frac{1}{10}}, \frac{1}{\frac{1}{100}}, \frac{1}{\frac{1}{1000}}, \frac{1}{\frac{1}{10000}}, \ldots\ldots$$

i.e. $$1, \ 10, \ 100, \ 1000, \ 10000, \ \ldots\ldots$$

i.e., $$1, \ 10, \ 10^2, \ 10^3, \ 10^4, \ \ldots\ldots, 10^n, \ldots\ldots$$

This has its general term as 10^n.

As $n \to \infty$, 10^n also becomes indefinitely large.

i.e., $10^n \to \infty$ as $n \to \infty$

i.e. $\underset{n \to \infty}{\text{Lt}} 10^n = \infty$.

Consider the function, $\frac{x^2 - 9}{x - 3}$

If $x \neq 3$, we can cancel out the factor $x - 3$, and the function $\frac{x^2 - 9}{x - 3}$ is the same as the function $x + 3$, since $x^2 - 9 = (x + 3)(x - 3)$. But when $x = 3$, we cannot cancel the factor $x - 3$ and $\frac{x^2 - 9}{x - 3}$ is of the form $\frac{0}{0}$ which is an indeterminate quantity. What happens when x approaches 3? Noticing the fact that x approaching 3 does not mean that x taking the value 3 we have the following table.

x	$f(x) = \frac{x^2 - 9}{x - 3} = x + 3$ when $x \neq 3$.
2·9	5·9
2·99	5·99
2·999	5·999
2·999......9	5·999......9
3	$\frac{0}{0}$
3·000......1	6·000......1
3·001	6·001
3·01	6·01
3·1	6·1

We see that $\frac{x^2 - 9}{x - 3}$ approaches 6 as x approaches 3 and also that the difference between $\frac{x^2 - 9}{x - 3}$ and 6 can be made as small as we please by taking x sufficiently closer to 3. Such a value 6 is called the limit of the function $\frac{x^2 - 9}{x - 3}$ as x approaches 3 and it is written symbolically as

$$\underset{x \to 3}{\text{Limit}} \frac{x^2 - 9}{x - 3} = 6 \quad \text{or} \quad \underset{x \to 3}{\text{Lt}} \frac{x^2 - 9}{x - 3} = 6.$$

Notice that the limit of the function as x approaches 3 is not the value of the function when $x = 3$. The limit is 6 whereas the value is indeterminate.

We can define the limit of a function in a general way as follows :

Definition. l is said to be the **limit** of the function $f(x)$ as x approaches a if the difference between l and $f(x)$ can be made as small as we please by taking x sufficiently nearer to a and in symbols, we write,

$$\underset{x \to a}{\text{Limit}} f(x) = l.$$

Limit of a function as x tends to a **need not** be the value of the function when $x = a$.

Example 1. Evaluate $\underset{x \to 0}{\text{Limit}}\ x^2 + 2x + 5$

As $x \to 0$, that is, as x becomes smaller and smaller x^2 and $2x$ become smaller and smaller. So the function $x^2 + 2x + 5$ becomes very close to 5. The difference between 5 and $x^2 + 2x + 5$ can be made as small as we please by taking x sufficiently small.

$$\therefore \lim_{x \to 0} (x^2 + 2x + 5) = 5.$$

Notice that the value of the function $x^2 + 2x + 5$ when $x = 0$ is 5 which is also the limit of the function $x^2 + 2x + 5$ as $x \to 0$.

Example 2. Evaluate $\lim_{x \to \infty} \frac{x+5}{2x+3}$

$$\lim_{x \to \infty} \frac{x+5}{2x+3} = \lim_{x \to \infty} \frac{\frac{x+5}{x}}{\frac{2x+3}{x}} = \lim_{x \to \infty} \frac{1 + \frac{5}{x}}{2 + \frac{3}{x}} = \frac{1}{2}$$

as $\frac{5}{x}, \frac{3}{x} \to 0$ as $x \to \infty$

Example 3. Evaluate $\lim_{n \to \infty} \frac{2n^2 + 3n + 5}{-5n^2 + 7n + 9}$

$$\lim_{n \to \infty} \frac{2n^2 + 3n + 5}{-5n^2 + 7n + 9} = \lim_{n \to \infty} \frac{\frac{2n^2 + 3n + 5}{n^2}}{\frac{-5n^2 + 7n + 9}{n^2}}$$

$$= \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{5}{n^2}}{-5 + \frac{7}{n} + \frac{9}{n^2}} = -\frac{2}{5}$$

Example 4. Evaluate $\lim_{n \to \infty} \frac{n}{n^2 + 3}$

$$\lim_{n \to \infty} \frac{n}{n^2 + 3} = \lim_{n \to \infty} \frac{1}{n + \frac{3}{n}} = 0.$$

as the denominator becomes indefinitely large as $n \to \infty$ because of the presence of n in the denominator.

Example 5. Evaluate $\lim_{x \to 0} \frac{5x^2 - 7x + 9}{2x^2 + 3}$

$$\lim_{x \to 0} \frac{5x^2 - 7x + 9}{2x^2 + 3} = \frac{9}{3} = 3.$$

as $5x^2, -7x, 2x^2 \to 0$ as $x \to 0$.

3·2 Continuity

Referring to table 1 given in the previous article we see that x can approach 3 in two ways. x can approach 3 through values less than 3 or through values greater than three. When we say the limit is 6 we mean, unless otherwise stated, that, whatever be the way by which x approaches 3 the function has the limiting value 6. Sometimes we may have different limits for the function for the two different approaches.

Definition. A function $f(x)$ is said to be **continuous** at $x = a$ if

$$\underset{x \to a}{\text{Limit}} \; f(x) = f(a) \text{ in whatever manner } x \to a.$$

Suppose the graph of a given function is as given in *Fig.* 1.

We find that when $x \to 4$ from the left the limit will be 1 and as $x \to 4$ from the right the limit will be 3. There is a jump at $x = 4$. Such a point is called a point of **discontinuity**. But the function is continuous at all the other values of x lying between 0 and 7.

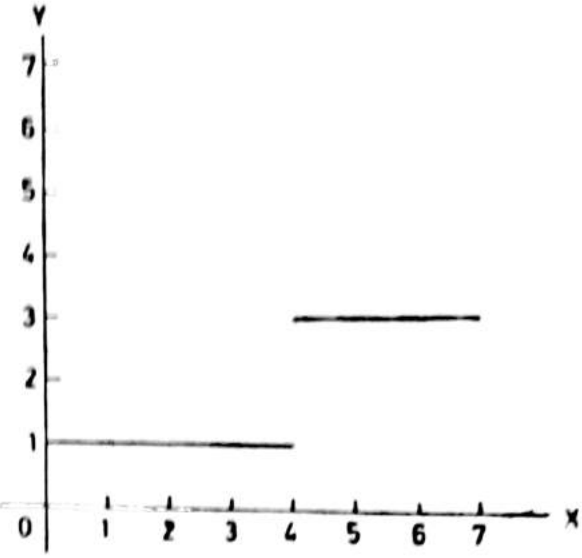


Fig. 1.

The collection of all values lying between 0 and 7 is called the interval 0 to 7 and is denoted by (0, 7). Of the two end points 0 and 7 either both or one among them or none of them may be included in the interval. If both end points are included in the interval it is called the closed interval $(0 \leq x \leq 7)$. If both points are excluded then the interval is called open interval $(0 < x < 7)$. If one of the end points alone is included then the interval is called semi-open.

We say that the function defined in (0, 7) is discontinuous at $x = 4$. A function is said to be **continuous** in an interval (a, b) if it is continuous for every value of x lying in the interval (a, b). We can say a continuous function defined over (a, b) is one whose graph can be drawn without taking the pen from the paper, that is to say that the graph of the function does not have gaps, breaks or jumps.

Discontinuous functions are not uncommon in occurrence. A firm may offer discounts on larger quantities purchased in order to attract large orders and thus achieve the economics of large production runs.

Example. A wholesale vegetable seller has the following price schedule :

Rs. 2.50 per kilogram for 20 kilograms and less

Rs. 2.00 per kilogram for more than 20 kgs. and not more than 50 kgs.

Rs. 1.75 per kilogram for more than 50 kgs. and not more than 100 kgs.

Rs. 1.50 per kilogram for more than 100 kgs.

If y is the price and x the quantity in kilograms, the price function is given by,

$$y = \begin{cases} 2.50\,x, & 0 \leqslant x \leqslant 20 \\ 2.00\,x, & 20 < x \leqslant 50 \\ 1.75\,x, & 50 < x \leqslant 100 \\ 1.50\,x, & x > 100 \end{cases}$$

The graph is given below.

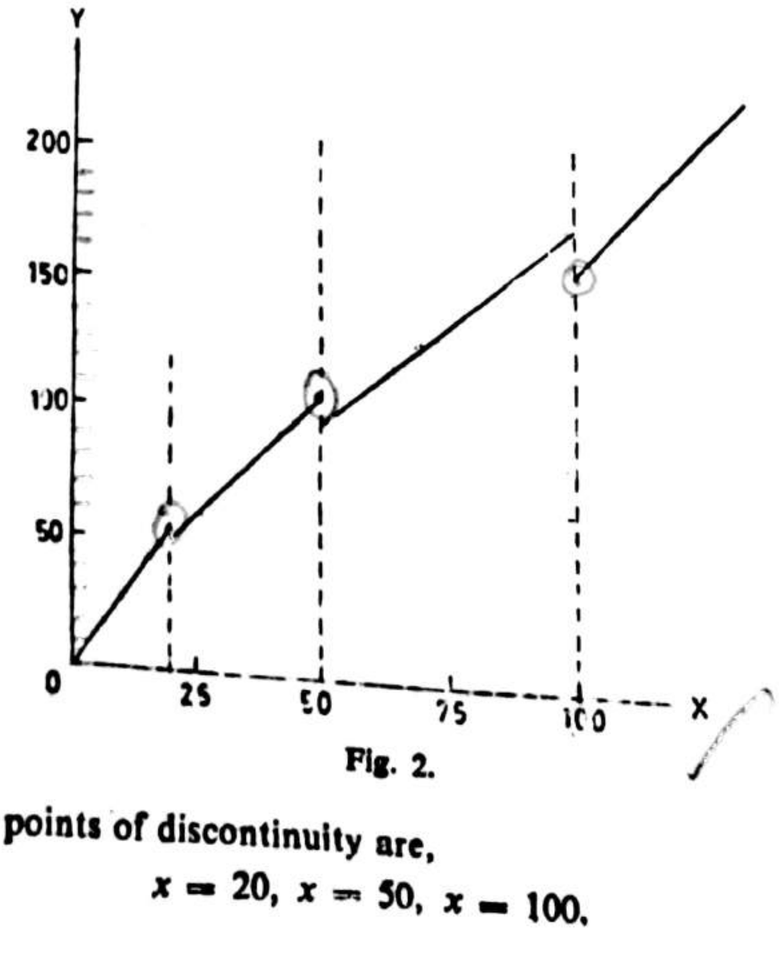


Fig. 2.

The points of discontinuity are,

$$x = 20,\ x = 50,\ x = 100.$$

(a) $\underset{x \to a}{\text{Limit}} \left\{ f(x) + g(x) \right\} = \underset{x \to a}{\text{Limit}}\, f(x) + \underset{x \to a}{\text{Limit}}\, g(x)$

(b) $\underset{x \to a}{\text{Limit}}\, f(x) \cdot g(x) = \underset{x \to a}{\text{Limit}}\, f(x) \cdot \underset{x \to a}{\text{Limit}}\, g(x)$.

(c) $\underset{x \to a}{\text{Limit}}\, \dfrac{f(x)}{g(x)} = \dfrac{\underset{x \to a}{\text{Limit}}\, f(x)}{\underset{x \to a}{\text{Limit}}\, g(x)}$ if $\underset{x \to a}{\text{Limit}}\, g(x) \neq 0$.

(d) $\underset{x \to a}{\text{Limit}} \left\{ f\{g(x)\} \right\} = f\left\{ \underset{x \to a}{\text{Limit}}\, g(x) \right\}$

(e) Express the above statements in words.

3·3. Changes in Related Variables

Calculus is the branch of Mathematics which deals with changes in related variables. It is the science of fluctuations. Naturally, therefore, Calculus has a role to play when we consider how the sales volume or sales is affected when the price changes or how the total cost, price etc. are affected when the volume of output changes and so on.

Notation. Δx, read as delta x, is used to denote the change in x or increment in x. It can be a positive change or a negative change. Δx, Δy, Δn are used to denote the changes in the variables x, y and n. "Δ" just stands for the words "change in the variable or quantity". Consider the function, $y = 2x + 5$. Let the initial value given to x be 1.

When $x = 1, \quad y = 7$

$x = 1 \cdot 5 \quad y = 8$

The change in x is $1 \cdot 5 - 1 = 0 \cdot 5$ *i.e.* $\Delta x = 0 \cdot 5$.

The corresponding amount by which y is affected is $8 - 7 = 1$.

Therefore $\Delta y = 1$.

So, for the function given above, when

$\Delta x = 0 \cdot 5, \quad \Delta y = 1.$

When $x = 1, \quad y = 7.$

When $x = 0 \cdot 9, \quad y = 2 \times 0 \cdot 9 + 5$

$= 1 \cdot 8 + 5 = 6 \cdot 8$

Now $\Delta x = 0 \cdot 9 - 1 = -0 \cdot 1$

and $\Delta y = 6 \cdot 8 - 7 = -0 \cdot 2$

As another example

Consider $y = -x^2$

When $x = 2, \quad y = -4$

$x = 1{\cdot}5, \quad y = -2{\cdot}25$

$\Delta x = 1{\cdot}5 - 2 = -0{\cdot}5$

$\Delta y = -2{\cdot}25 - (-4) = 1{\cdot}75.$

Example 1. Find Δy where $y = 2x+3$ if x is increased by the amount Δx.

$$y = 2x+3 \qquad \text{...(1)}$$

Let Δy be the amount by which y has increased when x is increased by Δx. So $y+\Delta y$ is the new value of the function when x is changed to $x+\Delta x$.

$$y+\Delta y = 2(x+\Delta x)+3 \qquad \text{...(2)}$$

(2)−(1) gives,

$$y+\Delta y-y = 2(x+\Delta x)+3-(2x+3)$$

$$\therefore \qquad \Delta y = 2\Delta x.$$

Example 2. Find Δy when $y = x^2$.

Arguing as in example (1),

$$y = x^2 \qquad \text{...(1)}$$

$$y+\Delta y = (x+\Delta x)^2 \qquad \text{...(2)}$$

(2)−(1) gives,

$$\Delta y = (x+\Delta x)^2 - x^2$$
$$= x^2+2x\Delta x+(\Delta x)^2-x^2$$
$$= 2x\,\Delta x+(\Delta x)^2.$$

Note. We notice that, here as $\Delta x \to 0$, $\Delta y \to 0$ *i.e.*, as the change in x tends to zero the change in y tends to zero. This is the property of any continuous function.

Example 3. Find the average rate of change in the function

$$y = 2x^2-5.$$

Let Δx be the increment in x and Δy the corresponding increment in y.

Average rate of change of y

$$= \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

Now $y = 2x^2-5 \qquad \text{...(1)}$

$$y+\Delta y = 2(x+\Delta x)^2-5$$
$$= 2\left[x^2+2x\Delta x+(\Delta x)^2\right]-5 \qquad \text{...(2)}$$

(2) − (1) gives

$$\Delta y = 4x\Delta x + 2(\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = \frac{4x\Delta x + 2(\Delta x)^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x.$$

What happens to the above ratio $\frac{\Delta y}{\Delta x}$, when x approaches zero ?

What is the interpretation of the ratio in this case ?

In this case the rate of change is not over a given interval, but at a particular value of x.

Problem Set 3 (*b*)

(1) Find Δy for a change Δx in x for the following functions $y = f(x)$ where

(*i*) $y = 2$. (*ii*) $y = 5x+7$.

(*iii*) $y = -2x+7$. (*vi*) $y = x^2+2x+7$.

(*v*) $y = x^3$. (*vi*) $y = \frac{1}{x}$.

(*vii*) $y = \sqrt{x}$. (*viii*) $y = \sqrt{2x+5}$.

(2) Find $\frac{\Delta y}{\Delta x}$ for the above. Do you notice any difference between the values got for (*i*), (*ii*) and (*iii*) and those got for (*iv*) to (*viii*).

(3) Find what happens when $\Delta x \to 0$. Can you interpret the results so obtained ?

3·4. Average Concept and Marginal Concept (Instantaneous Rate of Change)

The average concept expresses the variation of y over a whole range of values of x under consideration *i.e.*, from a given value to a certain selected value, say from x to $x+\Delta x$, to be particular say from 10 to 12 ($x = 10$, $\Delta x = 2$). Thus the average cost is the ratio between the total cost and the whole of the output concerned. Marginal concepts on the other hand express the variation on the "margin" *i.e.*, for every small variation of x from a given value of x. Therefore the marginal cost at a certain level of output is the change in the cost that results when the output is increased by a very small amount from the level. It is clear therefore that a marginal concept is precise only when it is considered in the limiting sense *i.e.*, as the variations in x are made smaller and smaller. Hence $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is interpreted as the marginal value of y at the level x.

As another example consider the average speed of the train over an hour's run and the speed of the train at the end of an hour's run. The first one is average concept and the second one is marginal concept. It is clear that the marginal change can also be considered as the instantaneous rate of change.

3·5. Differential Coefficient

Consider the function,

$$f(x) = \frac{x^2}{100}\ ,\ x \geqslant 0$$

where x is the investment in rupees and $y = f(x)$ is the total production of paddy in bags. The empirically meaningful part of the graph of the given function is shown in Fig. 3. Now the problem is, what is the marginal rate of paddy output at $x = 1000$ (Rupees). We cannot use $\frac{\Delta y}{\Delta x}$ over a given increment Δx to find the marginal rate at $x = 1000$, since marginal rate is change for any Δx no matter how small Δx may be. However we can approximate this marginal rate by computing $\frac{\Delta y}{\Delta x}$ for smaller and smaller values of Δx.

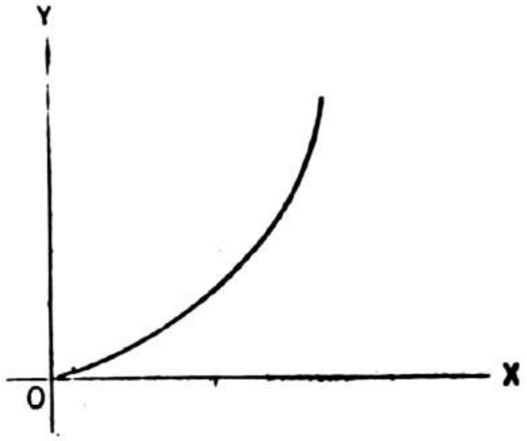


Fig. 3.

Let us consider the average rate of paddy out-put over a flexible increment from $x = 1000$ to $x = 1000 + \Delta x$.

$$f(1000) = \frac{1000^2}{100} = 10{,}000$$

$$f(1000+\Delta x) = \frac{(1000+\Delta x)^2}{100}$$

$$= 10{,}000 + 20\Delta x + \frac{(\Delta x)^2}{100}$$

$$y = f(1000+\Delta x) - f(1000)$$

$$= 20\Delta x + \frac{(\Delta x)^2}{100}$$

$$\frac{\Delta y}{\Delta x} = 20 + \frac{\Delta x}{100}\ ,\ \Delta x \neq 0.$$

Here $\frac{\triangle y}{\triangle x}$ gives the average rate of paddy out-put over an incremental investment $\triangle x$ begining at $x = 1000$ and ending at $x = 1000 + \triangle x$.

This increment is flexible in the sense that we can make its value equal to any quantity we wish by setting $\triangle x$ equal to that quantity.

Let $\triangle x = 500$,

then $\frac{\triangle y}{\triangle x} = 20 + \frac{500}{100} = 25$

Let $\triangle x = 200$.

then $\frac{\triangle y}{\triangle x} = 20 + \frac{200}{100} = 22$

Let $x = 100$

then $\frac{\triangle y}{\triangle x} = 20 + \frac{100}{100} = 21$.

Thus we have the following table,

$\triangle x$ in Rs.	500	200	100	50	10	0·1	0·01
$\frac{\triangle y}{\triangle x}$ in bags.	25	22	21	20·5	20·1	20·01	20·0001

We see that by taking values of $\triangle x$ near enough to zero we get values of the average rate of change as close to 20 bags as we wish.

So $\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = 20.$

20 bags is the exact marginal rate at $x = 1000$.

The above process is called the differentiation of the function $y = \frac{x^2}{100}$ with respect to the independent variable x at $x = 1000$.

Definition. Consider a function $y = f(x)$. Let $\triangle x$ be the increment given to x and let $\triangle y$ be the corresponding increment in y. Then the limit of the ratio $\frac{\triangle y}{\triangle x}$ as $\triangle x$ tends to zero, if it exists, is called the *Differential Coefficient or Derivative of y with respect to x* and the limit is denoted symbolically as $\frac{dy}{dx}$.

i.e., $$\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \frac{dy}{dx}$$

In other words, if $y = f(x)$ then

$$y+\triangle y = f(x+\triangle x)$$

$$\therefore \quad \triangle y = (y+\triangle y)-y$$

$$= f(x+\triangle x)-f(x)$$

$$\therefore \quad \frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$$

$$\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \frac{dy}{dx}$$

$$= \underset{\triangle x \to 0}{\text{Limit}} \frac{f(x+\triangle x)-f(x)}{\triangle x}$$

Note: 1. Various other notations for $\frac{dy}{dx}$: $f'(x)$ or y_1 or y' or Dy or $D f(x)$ where $D=\frac{d}{dx}$ and $y = f(x)$.

2. $\frac{d}{dx}$ stands for the statement "differentiate with respect to x" just like $\sqrt{\ }$ means taking the square root.

3. The process of finding the differential coefficient is called differentiation.

Example 1. Consider the function,

$$f(x) = x^2 \qquad \text{...(1)}$$

Let $\triangle x$ be the increment given to x and $\triangle y$ be the corresponding increment in $y = f(x)$.

Then $\quad f(x+\triangle x) = (x+\triangle x)^2$

$$= x^2+2x(\triangle x)+(\triangle x)^2 \qquad \text{...(2)}$$

(2)−(1) gives,

$$\triangle y = f(x+\triangle x)-f(x)$$

$$= 2x(\triangle x)+(\triangle x)^2$$

$$\frac{\triangle y}{\triangle x} = \frac{2x(\triangle x)+(\triangle x)^2}{\triangle x} = 2x+(\triangle x)$$

$$\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \underset{\triangle x \to 0}{\text{Limit}} (2x+\triangle x) = 2x.$$

This quantity $2x$ obtained is the differential coefficient of the function x^2 with respect to x.

$$\therefore \quad \frac{d}{dx}(x^2) = 2x.$$

or $\frac{dy}{dx} = 2x$ where $y = x^2$

or $f'(x) = 2x$ where $f(x) = x^2$

or $Dx^2 = 2x$

or $Dy = 2x$ where $y = x^2$

or $Df(x) = 2x$ where $f(x) = x^2$

Here D stands for $\frac{d}{dx}$.

3·6. Standard Forms

We give here some standard results. First we give the various stages in the process of finding the differential coefficient of a function using the definition of the differential coefficient alone.

Unless otherwise stated all the functions we consider will be continuous functions only.

The various stages are,

(i) Give an increment $\triangle x$ to the independent variable x.

(ii) Find the corresponding increment $\triangle y$ in y where $y = f(x)$.

(iii) Find $\frac{\triangle y}{\triangle x}$ (the average rate of change).

(iv) Evaluate $\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$ (the marginal or instantaneous rate of change).

Standard Form I

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ where } n \text{ is any constant.}$$

The proof of this result is not difficult.

For simplicity let us consider,

$$y = x^n$$

where n is any positive integer.

Then $y + \triangle y = (x + \triangle x)^n$

$\therefore \triangle y = (x + \triangle x)^n - x^n$

Using Binomial theorem,

$(x + \triangle x)^n = x^n + nx^{n-1}\triangle x + \frac{n(n-1)}{1 \cdot 2} x^{n-2}(\triangle x)^2 + \ldots\ldots\ldots$ terms containing higher powers of $\triangle x$.

$\therefore\quad (x+\triangle x)^n - x^n = nx^{n-1}\triangle +$ terms containing $(\triangle x)^2$ and higher powers of $\triangle x$.

$$\frac{(x+\triangle x)^n - x^n}{\triangle x} = nx^{n-1} + \text{terms containing } (\triangle x) \text{ and higher powers of } \triangle x.$$

$$\frac{\triangle y}{\triangle x} = nx^{n-1} + \text{terms containing } (\triangle x) \text{ and higher powers of } \triangle x.$$

$$\therefore\quad \underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = nx^{n-1}$$

This result is true when n is any constant and proof can be supplied by using the corresponding binomial theorem.

Example 1. Find $\frac{d}{dx}(\sqrt{x})$

$$\sqrt{x} = x^{1/2}$$

$$\frac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{1/2-1} = \frac{1}{2}x^{\frac{-1}{2}}$$

$$= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Example 2. Find $\frac{d}{dx}\left(\frac{1}{x^3}\right)$

We know that $\frac{1}{x^3} = x^{-3}$

$$\frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$$

Standard Form II

Find the differential coefficient of a constant with respect to x.

Let $y = c$, where c is any constant.

The problem is to find $\frac{dc}{dx}$ *i.e.* to find the instantaneous rate of change of a function which remains constant always. Obviously it is zero, *i.e.* $\frac{dc}{dx} = 0$.

It can be seen in the following way also.

Let $\quad y = c \qquad$ ...(1)

Then $\quad y + \triangle y = c \qquad$ ...(2)

since c is a constant not depending upon any variable.

(2)—(1) gives,

$$\triangle y = 0$$

$$\therefore \quad \frac{\triangle y}{\triangle x} = 0 \text{ as } \triangle x \neq 0$$

$$\therefore \quad \underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \underset{\triangle x \to 0}{\text{Limit}} \; 0 = 0.$$

Thus we have,

$$\frac{dc}{dx} = 0, \text{ where } c \text{ is any constant.}$$

Problem set 3(c)

Find the differential coefficient of the following functions :

(1) x^9 (2) $x^{3/2}$ (3) $\frac{1}{x}$ (4) $\frac{1}{\sqrt{x}}$

(5) $2x$ (6) $x^{-5/2}$ (7) $x^{\log 2}$ (8) 2^3.

Standard form III

Differential Coefficient of e^x

The function e^x is called the *Exponential Function.* e is an irrational number and it is a specific number which lies between 2·7 and 2·8. The function e^x arises in problems involving sales decay, present and future value of an investment in problems of finance, population growth etc. Once we have a new function one natural question is what is the differential coefficient.

Let $\qquad y=e^x$

and let $\triangle x$ be the increment given to x and let $\triangle y$ be the corresponding increment in y. ...(1)

Then $\qquad y+\triangle y=e^{x+\triangle x}$ ...(2)

(2)—(1) gives

$$\triangle y = e^{x+\triangle x} - e^x$$

$$= e^x(e^{\triangle x} - 1) \qquad \because a^{m+n} = a^m a^n$$

$$\frac{\triangle y}{\triangle x} = \frac{e^x(e^{\triangle x} - 1)}{\triangle x}$$

$$\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \underset{\triangle x \to 0}{\text{Limit}} \, e^x \frac{(e^{\triangle x} - 1)}{\triangle x}$$

$$= e^x \underset{\Delta x \to 0}{\text{Limit}} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] \text{ since } e^x \text{ does not involve } \Delta x$$

$$= e^x \times 1 = e^x$$

$$= e^x \times 1 \qquad \text{using the standard result}$$

$$= e^x \qquad \underset{h \to 0}{\text{Lt}} \frac{e^h - 1}{h} = 1$$

A Useful Rule in Differentiation

Consider a function $y = f(x)$. Let Δx be the increment in x and let Δy be the corresponding increment in y.

Clearly $$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$$

Taking the limit as $\Delta x \longrightarrow 0$

$$\underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta y}{\Delta x} = \frac{1}{\underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta x}{\Delta y}}$$

$$\frac{dy}{dx} = \frac{1}{\underset{\Delta y \to 0}{\text{Lt}} \frac{\Delta x}{\Delta y}},$$ since y is a continuous function of x

$$= \frac{1}{\frac{dx}{dy}}, \text{ by definition.}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

This rule will be very useful whenever we wish to find $\frac{dy}{dx}$ and finding $\frac{dx}{dy}$ is simpler than finding $\frac{dy}{dx}$.

Standard Form IV

Differential Coefficient of $\log_e x$

Let $$y = \log_e x \qquad ...(1)$$

The problem is to find $\frac{dy}{dx}$.

From (1) $x = e^y$ by the definition of logarithms. ...(2) (refer c of appendix I)

$$\therefore \quad \frac{dx}{dy} = e^y \qquad ...(3) \qquad \text{(Standard form)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y}, \quad \text{using (3)}$$

$$= \frac{1}{x} \text{ (using (2)).}$$

Note : 1. In calculus, unless otherwise stated, the base of logarithm is understood to be 'e'.

2. How to find the differential coefficient of $\log_a x$ when a is not necessarily the same as e? We use the change of base rule and standard form IV to find this. (refer c of appendix I)

The Technique of Differentiation

The process of finding the differential coefficients of functions can be systematised to enable the writing down of a differential coefficient an almost a mechanical affair. We evaluate differential coefficients only because they are useful to us in the application of Mathematical Methods to problems in Business. When evaluating differential coefficients using the definition alone we should find ourselves repeating over and over again the same kind of algebraic process. So we first evaluate the differential coefficients of simple standard functions, known as standard forms. The table of standard forms, once obtained is taken for granted. Then we formulate a set of rules which help us to differentiate complicated functions. A given function, however involved it may be, is reduced to a combination (such as sum, product, quotient etc.) of simpler functions for which we can get the differential coefficients from the table of standard forms. The rules of differentiation simply tell us how the differential coefficients of a combination of simple functions can be obtained. So with the table of standard forms and rules we can evaluate the differential coefficients with greater ease.

3·7. Rules of Differentiation

Sum Rule. Differential coefficient of the sum of two functions.

Let $$y = u + v \qquad ...(1)$$

where $u \equiv u(x)$, $v \equiv v(x)$ are functions of x.

For an increment $\triangle x$ in x, there will be changes in $u(x)$ and $v(x)$ and also in y as y is the sum of u and v. Let $\triangle u$, $\triangle v$ and $\triangle y$ be the corresponding increments.

Then $$y + \triangle y = u + \triangle u + v + \triangle v \qquad ...(2)$$

(2)−(1) gives

$$\triangle y = \triangle u + \triangle v$$

$$\frac{\triangle y}{\triangle x} = \frac{\triangle u + \triangle v}{\triangle x}$$

$$= \frac{\triangle u}{\triangle x} + \frac{\triangle v}{\triangle x}$$

$$\underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta y}{\Delta x} = \underset{x \to 0}{\text{Limit}} \left\{ \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right\}$$

$$= \underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta v}{\Delta x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

i.e. $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Note : (1) This can be extended to a sum of any finite number of functions.

$$\frac{d}{dx}(u+v+w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

(2) $$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

(3) $$\frac{d}{dx}(u+v-w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Example 1. Find the differential coefficient of x^2+x^3 with respect to x.

x^2+x^3 is the sum of two functions x^2 and x^3.

Let $$y = x^2+x^3$$

We know that,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2+x^3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$$

$$= 2x^{2-1} + 3x^{3-1}$$

$$= 2x + 3x^2.$$

Example 2. Find $\frac{d}{dx}\left(\frac{1}{x} + \sqrt{x} + e^x\right)$

$\frac{1}{x} + \sqrt{x} + e^x$ is the sum of three functions $\frac{1}{x}$, $\sqrt{x}$ and e^x

$$\frac{d}{dx}\left(\frac{1}{x} + \sqrt{x} + e^x\right) = \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(e^x)$$

$$= \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(e^x)$$

$$= (-1)x^{-1-1} + \frac{1}{2}x^{1/2-1} + e^x$$

$$= -x^{-2} + \tfrac{1}{2}x^{-1/2} + e^x$$

$$= -\frac{1}{x^2} + \frac{1}{2\sqrt{x}} + e^x$$

Example 3. Find the differential coefficient of $\frac{x^2+x^3+x^5}{x^2}$ with respect to x.

$$\text{Let } y = \frac{x^2+x^3+x^5}{x^2} = 1+x+x^3$$

$$\frac{dy}{dx} = \frac{d}{dx}(1+x+x^3) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^3)$$

$$= 0+1x^{1-1}+3x^{3-1}$$

$$= 0+1+3x^2 \qquad (\because x^0 = 1)$$

$$= 1+3x^2$$

Example 4. Find $\frac{d}{dx}\frac{(2x^2+x^{3/2}-x^5)}{x^2}$

$$\frac{d}{dx}\frac{(2x^2+x^{3/2}-x^5)}{x^2} = \frac{d}{dx}(2+x^{-1/2}-x^3)$$

$$= 0+(-\tfrac{1}{2})x^{-1/2-1}-3x^{3-1}$$

$$= -\frac{1}{2x^{3/2}}-3x^2$$

Example 5. Find $\frac{d}{dx}\left(\frac{x+x^2-x^3}{\sqrt{x}}+\log x\right)$

$$\frac{d}{dx}\left(\frac{x+x^2-x^3}{\sqrt{x}}+\log x\right) = \frac{d}{dx}(x^{1/2}+x^{3/2}-x^{5/2})+\frac{d}{dx}\log x$$

$$= \tfrac{1}{2}x^{-1/2}+\frac{3}{2}x^{1/2}-\frac{5}{2}x^{3/2}+\frac{1}{x}$$

$$= \frac{1}{2x^{1/2}}+\frac{3}{2}x^{1/2}-\frac{5}{2}x.x^{1/2}+\frac{1}{x}$$

$$= \frac{1}{2\sqrt{x}}+\frac{3\sqrt{x}}{2}-\frac{5}{2}x\sqrt{x}+\frac{1}{x}$$

Problem Set 3(d)

Find the differential coefficients of the following functions with respect to x.

(1) $x^7+x^{3/2}$.

(2) $x^{-7}+x^3-x$.

(3) $\frac{x^4+x^2+x}{x^2}$,

(4) $\frac{x^{5/2}-x^{3/2}+x^{1/2}}{x^{1/2}}$.

(5) $x^7+x^{3/2}+5+e^x+\log x$.

(6) $\frac{x^3-x^2-x}{\sqrt{x}}$.

(7) $\frac{x^2+1}{x^3}+\log x$.

Product Rule Differential coefficient of the product of two functions.

Let $$y = uv \qquad \ldots(1)$$
where $u = u(x)$, $v = v(x)$ be two functions of x.

Let $\triangle x$ be the increment given to x and let $\triangle u$, $\triangle v$, $\triangle y$ be the corresponding increments in u, v, y respectively.

Then,
$$y + \triangle y = (u + \triangle u)(v + \triangle v)$$
$$= uv + u\triangle v + v\triangle u + \triangle u \triangle v \qquad \ldots(2)$$

(2)−(1) gives

$$\triangle y = u\triangle v + v\triangle u + \triangle u \triangle v$$

$$\frac{\triangle y}{\triangle x} = u\frac{\triangle v}{\triangle x} + v\frac{\triangle u}{\triangle x} + \frac{\triangle u}{\triangle x}\triangle v$$

$$\frac{dy}{dx} = \underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \underset{\triangle x \to 0}{\text{Limit}} \left\{ u\frac{\triangle v}{\triangle x} + v\frac{\triangle u}{\triangle x} + \frac{\triangle u}{\triangle x}\triangle v \right\}$$

$$= \underset{\triangle x \to 0}{\text{Limit}}\ u\frac{\triangle v}{\triangle x} + \underset{\triangle x \to 0}{\text{Limit}}\ v\frac{\triangle u}{\triangle x} + \underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle u}{\triangle x}\triangle v$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}, \quad \text{since} \quad \underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle u}{\triangle x}\triangle v = 0$$

For, $\triangle v \to 0$ as $\triangle x \to 0$ since v is a continuous function and

$$\frac{du}{dx} \cdot 0 = 0$$

$$\therefore \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This rule is called the product rule of differentiation. This can be extended to the product of any finite number of functions. Notice that in the right hand side each term contains the differential coefficient of one function and the other function remains the same. No function is left "undifferentiated". When $y = uvw$ where u, v, w are functions of x,

$$\frac{dy}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$$

Note. Suppose $y = ku$ where k is a constant and u is a function of x.

Then, $\frac{dy}{dx} = k\frac{du}{dx} + u\frac{dk}{dx}$

$= k\frac{du}{dx}$ $\left(\because \frac{dk}{dx} = 0\right)$

Thus we have the useful result,

$$\frac{d}{dx}(ku) = k\frac{du}{dx} \text{ where } k \text{ is a constant.}$$

By this result,

$$\frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2\times 2x = 4x$$

$$\frac{d}{dx}(-4x^3) = -4\frac{d}{dx}(x^3) = -4\times 3x^2 = -12x^2$$

Example 1. Find the differential coefficient of the function $x^5.e^x$ w.r. to x. The given function is the product of two functions x^5 and e^x.

$$\frac{d}{dx}(x^5 e^x) = x^5\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^5)$$

$$= x^5 e^x + e^x(5x^4)$$

$$= x^5 e^x + 5x^4 e^x$$

$$= e^x(x^5 + 5x^4).$$

Example 2. Find $\frac{d}{dx}(2x^2 + 5\log x)(3x^4 + 7x^3)$

$$\frac{d}{dx}(2x^2 + 5\log x)(3x^4 + 7x^3)$$

$$= (2x^2 + 5\log x)\frac{d}{dx}(3x^4 + 7x^3) + (3x^4 + 7x^3)\frac{d}{dx}(2x^2 + 5\log x)$$

$$= (2x^2 + 5\log x)(3\times 4x^3 + 7\times 3x^2) + (3x^4 + 7x^3)\left(2\times 2x + \frac{5\times 1}{x}\right)$$

$$= (2x^2 + 5\log x)(12x^3 + 21x^2) + (3x^4 + 7x^3)\left(4x + \frac{5}{x}\right)$$

Example 3. Differentiate $e^x(\log x).(2x^2+3)$ with respect to x.

Let $y = e^x.\log x.(2x^2+3)$

$$\frac{dy}{dx} = e^x.\log x.\frac{d}{dx}(2x^2+3) + e^x(2x^2+3).\frac{d}{dx}(\log x)$$

$$+ (\log x).(2x^2+3)\frac{d}{dx}(e^x)$$

$$= e^x (\log x)(4x) + e^x (2x^2+3) . \frac{1}{x} + \log x . (2x^2+3) . e^x$$

$$= e^x \left[4x \log x + \frac{2x^2+3}{x} + (2x^2+3) \log x \right]$$

Example 4. Find $\frac{d}{dx}(\log_a x)$

We know that $\log_a x = \frac{\log_e x}{\log_e a}$

$$\therefore \frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)$$

$$= \frac{1}{\log_e a} . \frac{d}{dx}(\log_e x), \text{ Refer note under product rule}$$

$$= \frac{1}{\log_e a} . \frac{1}{x}$$

$$= \frac{1}{x \log a}$$

Problem Set 3(e)

Find the differential coefficients of the following functions with respect to x.

(1) (*a*) $\frac{2}{3}x$. (*b*) $-x^3$. (*c*) $\frac{1}{2}x^3$.

(*d*) $-4x^{-1/2}$. (*e*) $\frac{2}{x^3}$. (*f*) $\sqrt{4x}$.

(*g*) $-\frac{1}{3}x^{-6}$. (*h*) $\frac{1}{2x^3}$. (*i*) $2e^x$.

(*j*) $\frac{\log x}{5}$.

(2) (*a*) $x^5 e^x$. (*b*) $x^2 \log x$. (*c*) $e^x \log x$.

(*d*) $(x^3+1)(3x^2-2x^3)$. (*D.M.S. 1971*)

(*e*) $x^2 . e^x . \log x$.

(*f*) $(2x^3+3x^2+e^x)(7x^5-10+\log x)$

(*g*) $(-2x^3+7x-9)(5x^2-7x-12\sqrt{x})(x^2+3)$.

(*h*) $\left(x^2+\frac{1}{x}\right)\log x$.

(*i*) $\frac{\log x}{x^3}$. (*j*) $\frac{e^x}{x^{-3}}$,

(*k*) $(ax^2+bx+c)(hx^3+mx^2)$

where a, b, c, h, m, are constants.

Quotient Rule. Differential coefficient of the quotient of two functions.

Let y be the quotient of two functions $u(x)$ and $v(x)$.

i.e. $$y = \frac{u}{v} \qquad \text{...(1)}$$

Let $\triangle x$ be the increment given to x and $\triangle u$, $\triangle v$, $\triangle y$ be the corresponding increments in u, v, y respectively.

Then, $$y + \triangle y = \frac{u + \triangle u}{v + \triangle v} \qquad \text{...(2)}$$

(2)—(1) gives,

$$\triangle y = \frac{u + \triangle u}{v + \triangle v} - \frac{u}{v}$$

i.e., $$y = \frac{v(u + \triangle u) - u(v + \triangle v)}{(v + \triangle v)v}$$

$$= \frac{uv + v\triangle u - uv - u\triangle v}{v^2 + v\triangle v}$$

$$= \frac{v\triangle u - u\triangle v}{v^2 + v\triangle v}$$

$$\frac{\triangle y}{\triangle x} = \frac{v\dfrac{\triangle u}{\triangle x} - u\dfrac{\triangle v}{\triangle x}}{v^2 + v\triangle v}$$

$$\underset{\triangle x \to 0}{\text{Limit}} \frac{\triangle y}{\triangle x} = \underset{\triangle x \to 0}{\text{Limit}} \left\{ \frac{v\dfrac{\triangle u}{\triangle x} - u\dfrac{\triangle v}{\triangle x}}{v^2 + v\triangle v} \right\}$$

$$= \frac{\underset{\triangle x \to 0}{\text{Limit}} \left(v\dfrac{\triangle u}{\triangle x} - u\dfrac{\triangle v}{\triangle x} \right)}{\underset{\triangle x \to 0}{\text{Limit}} (v^2 + v\triangle v)}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

since $\triangle v \to 0$ as $\triangle x \to 0$ as $v(x)$ is a continuous function.

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

i.e. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\frac{(\text{Numerator})}{\text{Denominator}} = \frac{(D_r)\frac{d}{dx}(N_r) - N_r\frac{d}{dx}(N_r)}{(D_r)^2}$$

where $N_r \equiv$ Numerator and $D_r \equiv$ Denominator.

Example 1. Find $\frac{d}{dx}\left(\frac{x^3}{x^2+1}\right)$

$\frac{x^3}{x^2+1}$ is the quotient of the two functions x^3 and x^2+1.

$$\frac{d}{dx}\left(\frac{x^3}{x^2+1}\right) = \frac{(x^2+1)\frac{d}{dx}(x^3) - x^3\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\,3x^2 - x^3 \times 2x}{(x^2+1)^2}$$

$$= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2}$$

$$= \frac{3x^2+x^4}{(x^2+1)^2}$$

Example 2. Differentiate $\frac{x^4-9}{x^2+3}$ with respect to x.

(*B.B.A. April 1971*)

Let $y = \frac{x^4-9}{x^2+3}$

Applying the quotient rule,

$$\frac{dy}{dx} = \frac{(x^2+3)\frac{d}{dx}(x^4-9) - (x^4-9)\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)\,4x^3 - (x^4-9)\,2x}{(x^2+3)^2}$$

$$= \frac{4x^5+12x^3-2x^5+18x}{(x^2+3)^2}$$

$$= \frac{2x^5+12x^3+18x}{(x^2+3)^2}$$

Example 3. Differentiate $\frac{2x+3}{e^x+5}$ with respect to x.

Let $y = \frac{2x+3}{e^x+5}$

$$\frac{dy}{dx} = \frac{(e^x+5)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(e^x+5)}{(e^x+5)^2}$$

$$= \frac{(e^x + 5)(2) - (2x + 3)e^x}{(e^x + 5)^2}$$

$$= \frac{-e^x(1 + 2x) + 10}{(e^x + 5)^2}$$

Example 4. If $y = \dfrac{2 + 3 \log x}{x^2 + 5}$ find $\dfrac{dy}{dx}$

$$y = \frac{2 + 3 \log x}{x^2 + 5}$$

$$\frac{dy}{dx} = \frac{(x^2 + 5)\frac{d}{dx}(2 + 3 \log x) - (2 + 3 \log x)\frac{d}{dx}(x^2 + 5)}{(x^2 + 5)^2}$$

$$= \frac{(x^2 + 5)\left(\frac{3}{x}\right) - (2 + 3 \log x)\, 2x}{(x^2 + 5)^2}$$

$$= \frac{\frac{15}{x} - x - 6x \log x}{(x^2 + 5)^2}$$

Problem Set 3 (f)

1. Differentiate the following functions with respect to x:

(a) $\dfrac{2x + 1}{3x - 2}$ (b) $\dfrac{x^2 + 2}{x - 1}$ (c) $\dfrac{x^3}{2x - 1}$ (d) $\dfrac{4x^3 + 3x^2 + 11x}{7x^2 - 5x + 9}$

(e) $\dfrac{1}{2x + 5}$ (f) $\dfrac{1}{x^2 - 1}$ (g) $\dfrac{1}{2x^3 + 3x^2 - 12x + 9}$

(h) $\dfrac{7x + 6}{8x^2 + 4}$ (*DMS 1972*) (i) $\dfrac{2x + 3}{5 + 2e^x}$ (j) $\dfrac{5x - \log x}{2x^2 + 3x + 7}$

2. Show that,

$$\frac{d}{dx}\left\{\frac{k}{f(x)}\right\} = -\frac{k}{\left\{f(x)\right\}^2} f'(x) \text{ where } k \text{ is any constant.}$$

Function of a Function Rule or Chain Rule of Differentiation. This is one of the most important and useful rules.

Let x be land, u wheat and y bread. Assuming that for every unit of land (x) we can produce three units of wheat (u) we have,

$$u = 3x.$$

For every unit of wheat (u) we can produce 20 units of bread (y) and this will be given by,

$$y = 20u.$$

$$\log y = \log\Big[\quad (x - \ldots) \Big]$$

$$= \log\left[(x-1)^{3/2}(x-3)^{1/2}\right] - \log\left[(x-2)^2\right]$$

$$= \log (x-1)^{3/2} + \log (x-3)^{1/2} - \log (x-2)^2$$

$$\log y = \frac{3}{2} \log (x-1) + \frac{1}{2} \log (x-3) - 2 \log (x-2)$$

Differentiating both sides with respect to 'x'.

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{2(x-1)} + \frac{1}{2(x-3)} - \frac{2}{x-2}$$

$$\frac{dy}{dx} = y\left[\frac{3}{2(x-1)} + \frac{1}{2(x-3)} - \frac{2}{x-2}\right]$$

$$= \frac{(x-1)^{3/2}(x-3)^{1/2}}{(x-2)^2}\left[\frac{3}{2(x-1)} + \frac{1}{2(x-3)} - \frac{2}{x-2}\right]$$

Example 14. Find $\dfrac{d}{dx}\left[\dfrac{x^2+2}{e^{2x}+\log x}\right]$

$\dfrac{x^2+2}{e^{2x}+\log x}$ is the quotient of two functions x^2+2 and $e^{2x}+\log x$.

Hence,

$$\frac{d}{dx}\left[\frac{x^2+2}{e^{2x}+\log x}\right] = \frac{(e^{2x}+\log x)\frac{d}{dx}(x^2+2) - (x^2+2)\frac{d}{dx}(e^{2x}+\log x)}{(e^{2x}+\log x)^2}$$

$$= \frac{(e^{2x}+\log x)\,2x - (x^2+2)\left(2e^{2x} + \frac{1}{x}\right)}{(e^{2x}+\log x)^2}$$

Standard Form V.

To find the differential coefficient of a^x with respect to x.

Let $\quad y = a^x \quad \ldots(1) \quad$ To find $\dfrac{dy}{dx}$

Then $\quad x = \log_a y$ from the definition of logarithm.

$$x = \frac{\log_e y}{\log_e a} \qquad \ldots(2)$$

using change of base rule.

Differentiating both sides of (2) with respect to 'y'

We have, $$\frac{dx}{dy} = \frac{1}{\log_e a}\,\frac{d}{dy}(\log_e y)$$
$$= \frac{1}{y \log_e a}$$

But $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\frac{1}{y \log_e a}} = y \log_e a$$

$$\frac{dy}{dx} = a^x \log_e a \quad \text{from (1)}$$

Example 1. Find $\frac{dy}{dx}$ if $y = 4^x$

$$y = 4^x$$
$$\frac{dy}{dx} = 4^x \log_e 4$$

Example 2. Find $\frac{dy}{dx}$ if $y = 3^{mx}$

Take $mx = u$

Then $y = 3^u$, where $u = mx$

$$\frac{dy}{du} = 3^u \log_e 3, \qquad \frac{du}{dx} = m$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3^u \log_e 3 \times m$$
$$= 3^{mx} \log_e 3 \,.\, m$$
$$= m3^{mx} \log_e 3.$$

Alternatively,

$$y = 3^{mx} = (3^m)^x$$

$\therefore$ $$\frac{dy}{dx} = (3^m)^x \log_e 3^m$$
$$= 3^{mx} \log_e 3^m = m3^{mx} \log_e^3$$

Suppose $y = 5^{-x}$ then $\frac{dy}{dx} = 5^{-x} \log_e 5^{-1}$, since $m = -1$

$$= \frac{1}{5^x} \log_e \left(\frac{1}{5}\right)$$

Table I. Standard Forms

1. $\frac{dk}{dx} = 0$, where k is any constant.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is any constant.

3. $\frac{d}{dx} e^x = e^x$.

4. $\frac{d}{dx} \log_e x = \frac{1}{x}$.

5. $\frac{d}{dx} a^x = a^x \cdot \log_e a$.

Table II. Rules of differentiation.

1. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

2. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

3. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

4. $\frac{d}{dx}(ku) = k\frac{du}{dx}$, where k is any constant.

5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(Here u and v denote $u(x)$, $v(x)$, two functions of x.)

Table III. Chain rule and some useful results.

1. y is a function of u and u is a function of x.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2. $\frac{d}{dx}\{f(x)\}^n = n\{f(x)\}^{n-1} \cdot \frac{d}{dx}\{f(x)\}$

3. $\frac{d}{dx}\{e^{f(x)}\} = e^{f(x)} \cdot \frac{d}{dx}\{f(x)\}$

4. $\frac{d}{dx}\{\log f(x)\} = \frac{1}{f(x)} \cdot \frac{d}{dx}\{f(x)\}$

5. $\frac{d}{dx}\{a^{f(x)}\} = a^{f(x)} \log_e a \frac{d}{dx} f(x)$ (Prove)

3·10. Higher Order Derivatives

Consider the function,

$$y = x^5 \qquad \text{...(1)}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = 5x^4 \qquad \text{...(2)}$$

Since $5x^4$ is a function of x we can differentiate it with respect to x,

$$\frac{d}{dx}(5x^4) = 5 \times 4x^3 = 20x^3 \qquad \text{...(3)}$$

From (2) and (3), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 20x^3$$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is denoted by $\frac{d^2y}{dx^2}$ read as :

"d-squared y by dx squared". This is called the second order differential coefficient of y with respect to x. What we have done is precisely the differentiation of the function $y = x^5$ successively twice with respect to x.

We now give the various symbols used.

y_2, y'', $y^{(2)}$, $y''(x)$, $f^{(2)}(x)$, D^2y are the various symbols used to denote the second order derivative $\left(D^2 \equiv \frac{d^2}{dx^2}\right)$.

Similarly $\frac{d^3y}{dx^3}$ is the third order differential coefficient of y with respect to x and so on.

Concept of second and higher order derivatives are needed in optimization problems which we shall take up in the next chapter.

Example 1.

Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ if $y = 5x^3 - 7x$

$$y = 5x^3 - 7x$$

$$\frac{dy}{dx} = 15x^2 - 7$$

$$\frac{d^2y}{dx^2} = 30x$$

$$\frac{d^3y}{dx^3} = 30.$$

Example 2. If $y = x\ e^{x^2}$ find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$

$$y = xe^{x^2}$$

$$\frac{dy}{dx} = x.e^{x^2}2x + 1.e^{x^2}$$

$$= 2x^2e^{x^2} + e^{x^2}$$

$$= e^{x^2}(2x^2+1)$$

$$\frac{d^2y}{dx^2} = e^{x^2}.2x(2x^2+1) + e^{x^2}.4x$$

$$= e^{x^2}(4x^3+2x+4x)$$

$$= e^{x^2}(4x^3+6x)$$

$$\frac{d^3y}{dx^3} = e^{x^2}\ 2x(4x^3+6x) + e^{x^2}(12x^2+6)$$

$$= e^{x^2}(8x^4+12x^2+12x^2+6)$$

$$= e^{x^2}(8x^4+24x^2+6)$$

Problem Set 3 (i)

(1) Find $\frac{d^2y}{dx^2}$ for the following functions :

(*i*) $y = \log x.$ (*ii*) $y = x^4+3x^2+6x-2$

(*iii*) $y = 3x-7$ (*iv*) $y = 5x^3-6x$

(*v*) $y = 5\sqrt{x}$ (*vi*) $y = \frac{3}{\sqrt{x}}$

(*vii*) $y = x^2-5$ (*viii*) $y = xe^x$

(*ix*) $y = e^x \log x.$ (*x*) $y = x^2 \log x.$

(2) Find y_1, y_2, y_3, y_4 for the following functions :

(*i*) $y = x^2$ (*ii*) $y = x^3$

(*iii*) $y = 5x^4-7x^2$ (*iv*) $y = e^x$

(*v*) $y = \log x.$ (*vi*) $y = e^x \log x$

(*vii*) $y = \frac{10}{x}+5x-7$ (*viii*) $y = x^5 \log x.$

(*ix*) $y = e^{-2x}$ (*x*) $y = x^2 e^{-x^2}$

(3) Evaluate the second order derivatives of the functions given below at the specified values of the independent variable.

(*i*) $y = 2x^2-x^3$ at $x = 0$, and $x = \frac{4}{3}$

(*ii*) $y = 40-4x+x^2$ at $x = 2$

$$AN = LM = OM - OL$$
$$= (x+\triangle x) - x$$
$$= \triangle x.$$

$$BN = BM - NM = BM - AL = (y+\triangle y) - y = \triangle y.$$

Let us imagine that the point A is fixed and B moves along the curve towards A *i.e.* the line through A and B is rotating about A.

As B moves closer to A, $\triangle x$ becomes smaller. Finally when B coincides with A, the line through A and B takes the position AT and touches the curve at A. We have obtained the tangent AT as the limiting position of AB. As $\triangle x \to 0$, $B \to A$ and the line $AB \to AT$. Now what happens to the ratio $\dfrac{BN}{AN} = \dfrac{\triangle y}{\triangle x}$? We know that the ratio $\dfrac{\triangle y}{\triangle x}$ becomes $\dfrac{dy}{dx}$ as $\triangle x \to 0$. But $\dfrac{\triangle y}{\triangle x}$ is the slope of the straight line AB. The straight line AB becomes the tangent AT when $\triangle x \to 0$, and therefore $\dfrac{dy}{dx}$ will be the slope of AT, the tangent at A to the curve $y = f(x)$.

At this point we turn the reader's attention to article 1·3 (page 6) where we have mentioned about the variations in the slope depending upon the positions of the line. If the line is inclined to the right, the slope is positive, if the line is inclined to the left the slope is negative and if the line is parallel to the x-axis the slope is zero. The zero slope concept helps us in the determination of the maximum and minimum value of a function. For example we will be interested in the maximum value of profit functions and the minimum value of cost functions.

4·2. Increasing and decreasing functions

Let y denote the average cost and x the out-put and let the relation between them be,

$$y = 40 - 4x + x^2 \qquad \ldots(1)$$

Now the question is, does the average cost y increase, or decrease or remain stationary as the out-put x increases? We draw the meaningful portion of the curve representing the function (1). (Fig. 2.)

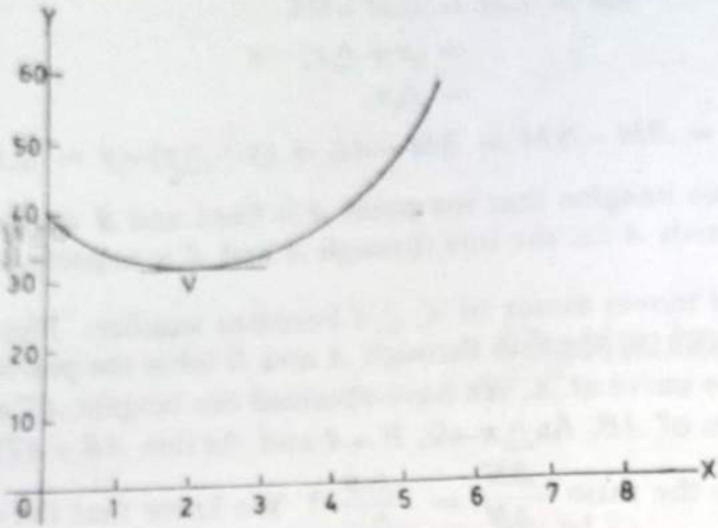


Fig. 2.

The graph shows that as the output x increases the average cost y decreases, reaches a minimum at V and then starts increasing.

Let us graph the part of the function (Fig. 3), which is to the left of V. The tangent at any point (excluding V) on this part is inclined to the left and the slope of such a tangent is negative.

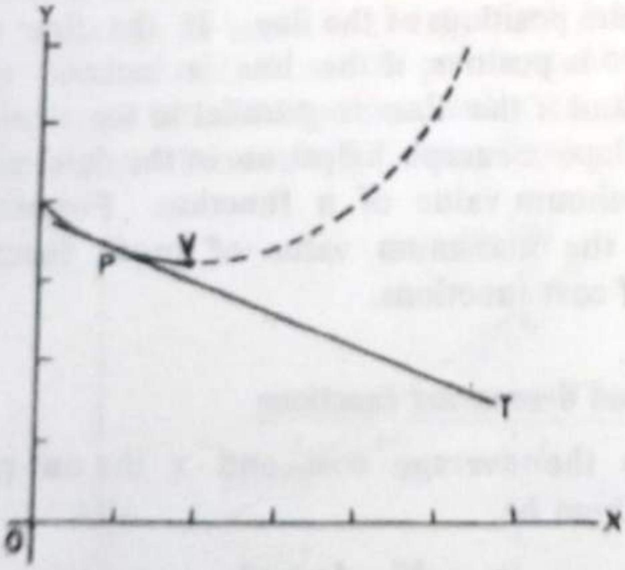


Fig. 3.

Let us graph the part of the function (Fig. 4) which is to the *right of V*. The tangent at any point (excluding V) on this part *is inclined to the right* and the slope of such a tangent is positive.

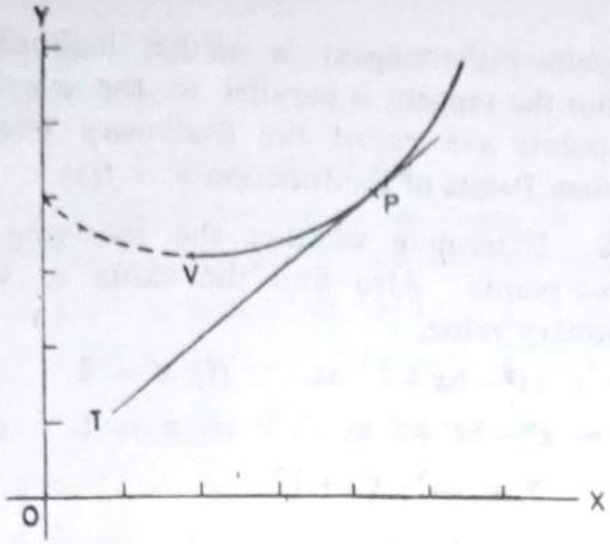


Fig. 4.

These facts can be verified by taking specific points on the curve like (1, 37), (4, 40) etc.

$$y = 40-4x+x^2$$

$$\frac{dy}{dx} = -4+2x$$

$$\left(\frac{dy}{dx}\right)_{(1,\ 37)} = -4+2 = -2 < 0.$$

$$\left(\frac{dy}{dx}\right)_{(4,\ 40)} = -4+8 = \quad 4 > 0.$$

Thus we say.

(*i*) If $y = f(x)$ decreases as x increases at the point (x, y) then $\frac{dy}{dx} < 0$ at (x, y) and conversely.

(*ii*) If $y = f(x)$ increases as x increases at the point (x, y) then $\frac{dy}{dx} > 0$ at (x, y) and conversely.

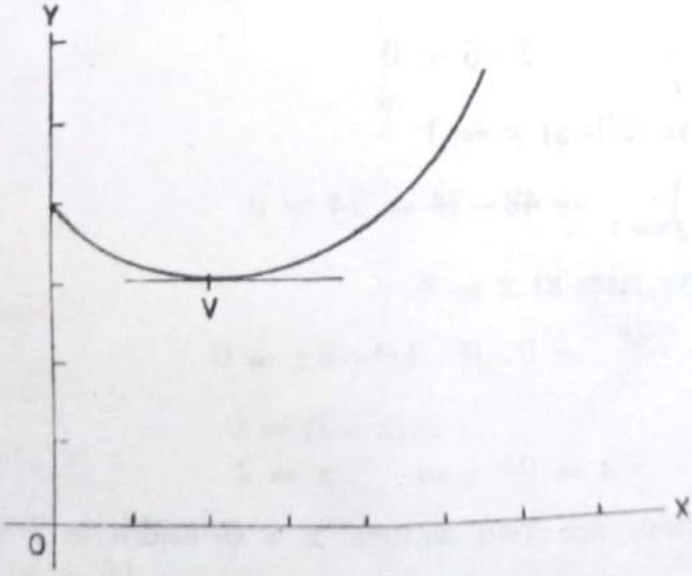


Fig. 5.

At the point V the tangent is neither inclined to the right nor to the left. But the tangent is parallel to the x-axis and its slope is zero. Such points are called the **Stationary Points** or **Extreme Points** or **Turning Points** of the function $y = f(x)$.

Example. Determine whether the following curves rise or fall at the given points. Also find the value or values of x which gives y a stationary value.

(1) $y = 2x^2 - 6x + 2$ at (*i*) $x = 2$ (*ii*) $x = 1$.

(2) $y = x^3 - 3x^2 + 5$ at (*i*) $x = 1$ (*ii*) $x = 4$.

(1) $$y = 2x^2 - 6x + 12$$

$$\frac{dy}{dx} = 4x - 6.$$

At $x = 2$, $\frac{dy}{dx} = 8 - 6 = 2 > 0$

i.e., y increases as x increases from 2.

i.e., the curve rises at $x = 2$

At $x = 1$, $\frac{dy}{dx} = 4 - 6 = -2 < 0$

i.e. y decreases as x increases from 1.

i.e. the curve falls at $x = 1$.

Now $\frac{dy}{dx} = 0$ if $4x - 6 = 0$ *i.e.* at $x = \frac{6}{4} = \frac{3}{2}$

$\therefore$ $x = \frac{3}{2}$ gives a stationary value to y.

(2) $$y = x^3 - 3x^2 + 5$$

$$\frac{dy}{dx} = 3x^2 - 6x.$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 3 - 6 < 0$$

The curve falls at $x = 1$

$$\left(\frac{dy}{dx}\right)_{x=4} = 48 - 24 = 24 > 0$$

The curve rises at $x = 4$.

$$\frac{dy}{dx} = 0 \text{ if } 3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

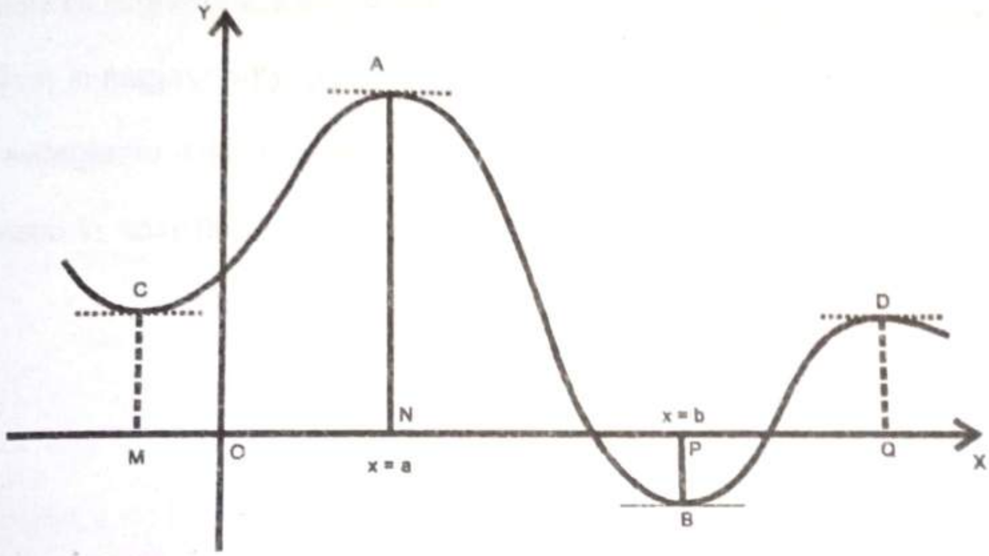
i.e. $x = 0$ or $x = 2$

Thus there are two values $x = 0$ and $x = 2$ which make y stationary.

6
MAXIMA AND MINIMA

Definition : **A function $y = f(x)$ is said to have attained its Local Maximum value at $x = a$ if the function ceases to increase and begins to decrease at $x = a$, in the neighbourhood of $x = a$**

Definition : **A function $y = f(x)$ is said to have attained its Local Minimum value at $x = b$ if the function ceases to decrease and begins to increase at $x = b$, in the neighbourhood of $x = b$.**



Suppose let us consider the graph of a function of x as given above. The points A and D are called the **Local Maximum** points of the graph, and the points B and C are called the **Local Minimum** points. The values AN and DQ are called local maximum values at A and D respectively. The values CM and BP are called local minimum values at C and B respectivelv.

Note : It should be carefully noted that according to the definition given above, it is clear that

(i) the **'local maximum'** and the **'local minimum'** values of a function **does not mean** the **'greatest'** and the **'least' values of the function.**

(ii) the function may **have several local maximum and local minimum values.**

(iii) in the shape of the curve, the **maxima are like maintain tops** and the **minima like valley bottoms**.

(iv) the points at which a function have a maximum or minimum values are called **turning points.**

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(v) The maxima and minima points of a function are also called as extreme points of the function.

Criteria for maxima and minima

(a) Suppose a function $y = f(x)$ has a local maximum at $x = a$. Then by the definition, it is an increasing function for values x which just precede 'a' and is a decreasing function for value of x which just follow 'a'. That is, its derivative $\frac{dy}{dx}$ is positive, for values just before 'a' and negative, just after 'a'. Thus at $x = a$, it changes its sign from positive to negative and it is zero. Hence $\frac{dy}{dx} = 0$ at $x = a$ (the tangent is parallel to x -axis at $x = a$). As well, the value of $\frac{dy}{dx}$ is changing from positive to negative in the neighbourhood of 'a' and $\frac{dy}{dx}$ in a decreasing function. By the definition of decreasing function, its derivative, $\frac{d^2y}{dx^2}$ is < 0.

Hence for maxima, **(i)** $\frac{dy}{dx} = 0$ and **(ii)** $\frac{d^2y}{dx^2} < 0$.

(b) Suppose a function $y = f(x)$ has a local minimum at $x = b$. Then by definition, it is a decreasing function for values of x which just precede 'b' and is an increasing function for values of x which just follow 'b'. That is, its derivative $\frac{dy}{dx}$ is negative, for values just before $x = b$ and positive, just after 'b'. Hence $\frac{dy}{dx} = 0$ at $x = b$ (the tangent is parallel to x - axis at $x = b$). Also, the value of $\frac{dy}{dx}$ is changing from negative to positive in the neighbourhood of 'b' and $\frac{dy}{dx}$ is an increasing function. By the definition of increasing function, its derivative $\frac{d^2y}{dx^2}$ is >0.

Hence for minima, **(i)** $\frac{dy}{dx} = 0$ and **(ii)** $\frac{d^2y}{dx^2} > 0$.

Working Rule for finding maximum and minimum values of a function

Step I : Find $\frac{dy}{dx}$ for the given function $y = f(x)$

Step II : Find the values of x, say a, b, c,. . . which make $\frac{dy}{dx} = 0$

Step III : Find $\frac{d^2y}{dx^2}$

Step IV : Put $x = a$ in $\frac{d^2y}{dx^2}$. If the value is negative, the function has a maximum at $x = a$ and maximum of $y = f(a)$. If the value is positive, the function has a minimum at $x = a$ and minimum of $y = f(a)$. Similarly test for other values b, c, . . . of x found in step II.

Step V : When $\frac{d^2y}{dx^2} = 0$ for a particular value $x = a$ (say) then find $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$ and put $x = a$ successively in these derivatives.

MISCELLANEOUS ILLUSTRATION :

Illustration 1 : Find the maximum and minimum values of $x^3 - 3x^2 - 9$.

Solution : Let $y = x^3 - 3x^2 - 9$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x^2 - 9)$$

$$= \frac{d}{dx}(x^3) - 3\cdot\frac{d}{dx}(x^2) - \frac{d}{dx}(9) = 3x^{3-1} - 3\cdot 2x^{2-1} - 0 = 3x^2 - 6x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 6x) = 3\frac{d}{dx}(x^2) - 6\cdot\frac{d}{dx}(x) = 3\cdot 2x^{2-1} - 6\cdot 1 = 6x - 6$$

For maximum or minimum, $\frac{dy}{dx} = 0$

i.e. $3x^2 - 6x = 0$

$3x(x - 2) = 0$

$$3x = 0 \qquad x - 2 = 0$$
$$\therefore x = 0 \qquad \therefore x = 2.$$

When $x = 0$, $\frac{d^2y}{dx^2} = 6(0) - 6 = -6$ (negative).

$\therefore$ The function has maximum at $x = 0$ since $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

$\therefore$ The maximum value of the function $= (0)^3 - 3(0)^2 - 9 = -9$

When $x = 2$, $\frac{d^2y}{dx^2} = 6(2) - 6 = 12 - 6 = 6$ (positive).

$\therefore$ The function has minimum at $x = 2$, since $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

$\therefore$ The minimum value of the function $= (2)^3 - 3(2)^2 - 9$

$$= 8 - 12 - 9 = -13.$$

Illustration 2 : Find the maximum and minimum values of $2x^3 + 3x^2 - 12x - 6$.

Solution : Let $y = 2x^3 + 3x^2 - 12x - 6$.

$$\frac{dy}{dx} = 2\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) - 12\frac{d}{dx}(x) - \frac{d}{dx}(6)$$

$$= 2 \cdot 3x^{3-1} + 3 \cdot 2x^{2-1} - 12 \cdot 1 - 0 = 6x^2 + 6x - 12$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x^2 + 6x - 12)$$

$$\frac{d^2y}{dx^2} = 6 \cdot \frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) - \frac{d}{dx}(12) = 6 \cdot 2x^{2-1} + 6 \cdot 1 - 0 = 12x +$$

For minimum or maximum, $\frac{dy}{dx} = 0$

$$6x^2 + 6x - 12 = 0$$

$$6(x^2 + x - 2) = 0, \text{ since } 6 \neq 0, \quad x^2 + x - 2 = 0$$

$$x^2 + 2x - x - 2 = 0$$

$$x(x + 2) - 1(x + 2) = 0$$

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$$(x-1)(x+2) = 0$$

$x - 1 = 0 \qquad x + 2 = 0$

$\therefore x = 1 \qquad \therefore x = -2$

When $x = 1$, $\dfrac{d^2y}{dx^2} = 12(1) + 6 = 18$ (positive)

$\therefore$ The function has minimum value at $x = 1$ $\left[\because \dfrac{dy}{dx} = 0 \text{ and } \dfrac{d^2y}{dx^2} > 0\right]$

$\therefore$ The minimum value of the function $= 2\cdot(1)^3 + 3(1)^2 - 12(1) - 6 = -13$

When $x = -2$, $\dfrac{d^2y}{dx^2} = 12(-2) + 6 = -24 + 6 = -18$ (negative).

$\therefore$ The function has maximum value at $x = -2$, $\left[\because \dfrac{dy}{dx} = 0 \text{ and } \dfrac{d^2y}{dx^2} < 0\right]$

$\therefore$ The maximum value of the function $= 2(-2)^3 + 3(-2)^2 - 12(-2) - 6$

$= -16 + 12 + 24 - 6 = 14.$

Illustration 3 : Investigate the maxima and minima of the function

$3x^4 + 16x^3 + 18x^2 + 20.$

Solution : Let $y = 3x^4 + 16x^3 + 18x^2 + 20$

$$\frac{dy}{dx} = 3\cdot\frac{d}{dx}\left(x^4\right) + 16\cdot\frac{d}{dx}\left(x^3\right) + 18\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(20)$$

$$= 3\cdot 4x^{4-1} + 16\cdot 3x^{3-1} + 18\cdot 2x^{2-1} + 0 = 12x^3 + 48x^2 + 36x$$

$$\frac{d^2y}{dx^2} = 12\cdot\frac{d}{dx}\left(x^3\right) + 48\frac{d}{dx}\left(x^2\right) + 36\frac{d}{dx}(x)$$

$$= 12\cdot 3x^{3-1} + 48\cdot 2x^{2-1} + 36\cdot 1 = 36x^2 + 96x + 36.$$

For Maximum or Minimum, $\dfrac{dy}{dx} = 0$

i.e. $\qquad 12x^3 + 48x^2 + 36x = 0$

$$12x(x^2 + 4x + 3) = 0$$

$$12x(x^2 + 3x + x + 3) = 0; \quad 12x = 0 \quad \therefore x = 0.$$

$$x(x+3) + 1(x+3) = 0$$

$$(x+1) + (x+3) = 0$$

$$(x+1) = 0 \qquad (x+3) = 0$$

$$\therefore x = -1; \qquad x = -3; \qquad x = 0.$$

When $x = -1$, $\frac{d^2y}{dx^2} = 36(-1)^2 + 96(-1) + 36$

$$= 36 - 96 + 30 = -30 \text{ (negative).}$$

$\therefore$ The function has maximum value at $x = -1$ $\left[\because \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0\right]$

$\therefore$ The maximum value of the function $= 3(-1)^4 + 16(-1)^3 + 18(-1)^2 + 20$

$$= 3 - 16 + 18 + 20 = 25.$$

When $x = -3$, $\frac{d^2y}{dx^2} = 36(-3)^2 + 96(-3) + 36 = 324 - 288 + 36 = 72$ (positive)

$\therefore$ The function has minimum value at $x = -3$ $\left[\because \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0\right]$

$\therefore$ The minimum value at $x = -3 = 3(-3)^4 + 16(-3)^3 + 18(-3)^2 + 20$

$$= 243 - 432 + 162 + 20 = -7$$

When $x = 0$, $\frac{d^2y}{dx^2} = 36(0)^2 + 96(0) + 36 = 36$ (positive).

$\therefore$ The function has minimum value at $x = 0$.

The minimum value at $x = 0 = 3(0)^4 + 16(0)^3 + 18(0)^2 + 20 = 20.$

may be the highest (lowest) point in its immediate neighbourhood but there may be higher (lower) points on the curve outside the neighbourhood.

Step by Step Procedure to find the Maxima and Minima of a Function of one Variable

(1) Find $\frac{dy}{dx}$ for the given function $y = f(x)$.

(2) Find the value or values of x which make $\frac{dy}{dx}$ zero. Let these be $x_1, x_2, x_3, \ldots\ldots$

(3) Find $\frac{d^2y}{dx^2}$.

(4) Find the sign of $\frac{d^2y}{dx^2}$ at $x = x_1, x_2, \ldots\ldots$ and hence decide which of these $x_1, x_2, \ldots\ldots$ maximise or minimise the function.

(5) Find the maximum and minimum values of the function by substituting for x in $y = f(x)$ by choosing the suitable value of x from $x_1, x_2, x_3, \ldots\ldots$

Sometimes, by the very nature of the problem, it will be clear whether the value of x which makes $\frac{dy}{dx}$ zero, maximises y or minimises y. In such cases there is no need for the second derivative test.

We tabulate the result as :

	Maximum	*Minimum*
Necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient condition	$\frac{dy}{dx} = 0;\ \frac{d^2y}{dx^2} < 0,$	$\frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} > 0$

In Economics $\frac{dy}{dx} = 0$ is called the first order condition and $\frac{d^2y}{dx^2} \lessgtr 0$ is called the second order condition for the occurrence of extreme values.

Example 1. Examine the cost function, $y = 40 - 4x + x^2$ for maximum or minimum.

$$y = 40 - 4x + x^2 \qquad \ldots(1)$$

$$\frac{dy}{dx} = -4 + 2x \qquad \ldots(2)$$

$$\frac{dy}{dx} = 0 \quad \text{if} \quad -4+2x = 0$$
$$x = 2$$

Differentiating (2), with respect to x,

$$\frac{d^2y}{dx^2} = 2 > 0 \text{ always}$$

$\therefore$ $x = 2$ makes y, a minimum and this minimum value of y is got by putting $x = 2$ in (1).

$$y = 40-4\times 2+2^2 = 40-8+4 = 36.$$

Example 2. Examine the function $y = 2x^2 - x^3 + 5$ for maximum and minimum.

$$y = 2x^2 - x^3 + 5 \quad \text{...(1)}$$

$$\frac{dy}{dx} = 4x - 3x^2 \quad \text{...(2)}$$

$$\frac{dy}{dx} = 0 \text{ gives, } \quad 4x - 3x^2 = 0$$

$$\text{i.e. } x(4-3x) = 0$$

$$x = 0 \quad \text{or} \quad \frac{4}{3}$$

Thus $x = 0$, and $x = \frac{4}{3}$ are the values of x which make y an extremum. (These points are also called critical points).

Differentiating (2) with respect to x,

$$\frac{d^2y}{dx^2} = 4 - 6x$$

At $x = 0$, $\quad \frac{d^2y}{dx^2} = 4 - 0 = 4 > 0$

$\therefore$ $x = 0$ makes y a minimum and putting $x = 0$ in (1) minimum value of y is 5.

At $x = \frac{4}{3}$, $\quad \frac{d^2y}{dx^2} = 4 - 8 = -4 < 0$

$\therefore$ $x = \frac{4}{3}$ makes y a maximum and the maximum value is obtained from (1) by putting $x = \frac{4}{3}$.

$$y = 2\left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^3 + 5 = \frac{32}{9} - \frac{64}{27} + 5$$

$$= \frac{32}{27} + 5$$

$$= \frac{167}{27}$$

Example 3. Examine for extrema, if any, for the function $y = x^5+5x^3+6$. (x is real).

$$y = x^5+5x^3+6$$

$$\frac{dy}{dx} = 5x^4+15x^2$$

$$\frac{dy}{dx} = 0 \text{ gives}$$

$$5x^4+15x^2 = 0$$

$$5x^2(x^2+3) = 0$$

$$x = 0.$$

Since x is real

$$x^2+3 \neq 0$$

$$\frac{d^2y}{dx^2} = 20x^3+30x$$

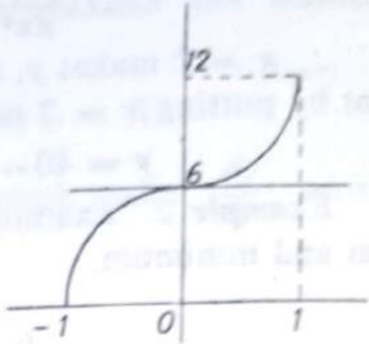

Fig. 10.

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = x\,(20x^2+30)$ and is less than zero if x is less than zero.

Also $\frac{d^2y}{dx^2}$ is $0 >$ if $x > 0$.

Hence $\frac{d^2y}{dx^2}$ changes sign when it passes through $x = 0$.

Hence $x = 0$ is a point of inflexion. The graph of

$$y = x^5+5x^3+6$$

is given in Fig. 10.

Problem Set 4(b)

1. Find the turning points of the following functions and determine whether they make the function maximum or minimum.

(*a*) $y = 27-x^3$.

(*b*) $y = x+x^{-1}$

(*c*) $y = k^{-x^2}$

(*d*) $y = 2\,x^3e^{-x}$.

(*e*) $y = kxe^{-x^2}$

(*f*) $y = x^3-6x$

(*g*) $y = (2-x)^2+(3-x)^2+(5-x)^2$

Note : 1. Is the second derivative test necessary to prove that $r = \left(\frac{25}{\pi}\right)^{1/3}$ minimises S ? Is it not obvious from practical considerations ?

2. If the box has a lid,

$$S = 2\pi rh + 2\pi r^2$$

Example 3. A box with square top and bottom is to be made to contain 250 cubic cms. Material for top and bottom costs Rs. 2 per square cm. and the material for the side costs Re. 1 per square cm. What is the cost of the least expensive box that can be made ?

As the base of the box is a square, the dimensions can be taken as x, x, y. Then the volume is

$$x^2y \text{ and } x^2y = 250 \quad \ldots(1)$$

Let C be the total cost.

The area of the top and bottom

$$= 2x^2 \text{ sq. cm.}$$

Cost for the top and bottom

$$= \text{Rs. } 2 \times 2x^2 = \text{Rs. } 4x^2$$

The area of the 4 lateral sides

$$= 4xy \text{ sq. cm.}$$

Cost for the 4 sides $= \text{Rs. } 1 \times 4xy$

$$= \text{Rs. } 4xy$$

$$C = 4x^2 + 4xy \quad \ldots(2)$$

We have to minimise C in (2) subject to the condition $x^2y = 250$. The expression C can be made as a function of single variable using the condition (1).

From (1), we have

$$y = \frac{250}{x^2}$$

Using this in (2),

$$C = 4x^2 + 4x\,\frac{250}{x^2}$$

$$C = 4x^2 + \frac{1000}{x} \quad \ldots(3)$$

$$\frac{dC}{dx} = 8x - \frac{1000}{x^2}$$

$\frac{dC}{dx} = 0$ gives,

$$8x - \frac{1000}{x^2} = 0$$

$$8x^3 = 1000$$

$$x^3 = 125$$

$$x = 5$$

$$\frac{d^2\overline{C}}{dx^2} = 8 + \frac{2 \times 1000}{x^3} > 0 \quad \text{when} \quad x = 5.$$

Thus $x = 5$ minimises C.

Substituting $x=5$ in (3)

$$C_{MIN} = 4 \times 25 + \frac{1000}{5}$$

$$= 100 + 200$$

$$= \text{Rs. } 300$$

Example 5. The production manager of a company plans to include 180 square centimetres of actual printed matter in each page of a book under production. Each page should have a 2·5 cm. margin along the top and bottom and 2·0 cm wide margin along the sides. What are the most economical dimensions of each printed page ?

Let x, y be the length and breadth of the printed matter in each page. Then,

$$xy = \text{area of the printed matter}$$

$$xy = 180 \qquad \ldots(1)$$

After taking into consideration the space proved for the margin the dimensions of each page are $x+4$ and $y+5$.

Let S be the area of each page. Then,

$$S = (x+4)(y+5) \qquad \ldots(2)$$

We have to minimise S subject to the constraint (1).

Now
$$S = xy+5x+4y+20$$
$$= 180+5x+4\times\frac{180}{x}+20 \qquad \text{from (1)}$$
$$= 5x+\frac{720}{x}+200$$
$$\frac{dS}{dx} = 5-\frac{720}{x^2}$$
$$\frac{dS}{dx} = 0 \text{ gives, } 5-\frac{720}{x^2} = 0$$

i.e.,
$$5x^2-720 = 0$$
$$x^2 = 144 = 12^2$$
$$x = 12, \text{ disregarding the negative value.}$$
$$\frac{d^2S}{dx^2} = \frac{2\times 720}{x^3} > 0, \text{ when } x = 12.$$

Therefore $x = 12$ minimises S.

Putting $x = 12$ in (1), $y = \frac{180}{12} = 15$

Thus the most economical dimensions are,

Length $= x+4 = 16$ cms.

Breadth $= y+5 = 20$ cms.

Illustration 4 : Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor a minimum value on the real line.

Solution : Let $y = x^3 - 3x^2 + 3x + 7$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x) + \frac{d}{dx}(7)$$

$$= 3 \cdot x^{3-1} - 3 \cdot 2x^{2-1} + 3 \cdot 1 + 0 = 3x^2 - 6x + 3.$$

$$\frac{d^2y}{dx^2} = 3 \cdot \frac{d}{dx}(x^2) - 6 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(3)$$

$$= 3 \cdot 2x^{2-1} - 6 \cdot 1 + 0 = 6x - 6.$$

For maximum or minimum, $\frac{dy}{dx} = 0$

i.e. $3x^2 - 6x + 3 = 0$

$3(x^2 - 2x + 1) = 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$x^2 - x - x + 1 = 0$

$x(x-1) - 1(x-1) = 0$

$(x-1)(x-1) = 0$

$x - 1 = 0, \quad x - 1 = 0$

$\therefore x = 1 \quad \therefore x = 1.$

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 6 = 0$ (neither positive nor negative)

$\therefore$ The function has neither maximum nor minimum.

Note : $x = 1$ is a point of inflexion.

Illustration 5 : Show that $x^5 - 5x^4 + 5x^3 - 30$ has a maximum when $x = 1$, a minimum when $x = 3$ and neither when $x = 0$.

Solution : Let $y = x^5 - 5x^4 + 5x^3 - 30$

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) - 5\frac{d}{dx}(x^4) + 5\frac{d}{dx}(x^3) - \frac{d}{dx}(30)$$

$$= 5x^{5-1} - 5\cdot 4x^{4-1} + 5\cdot 3x^{3-1} - 0 = 5x^4 - 20x^3 + 15x^2.$$

$$\frac{d^2y}{dx^2} = 5\cdot\frac{d}{dx}(x^4) - 20\frac{d}{dx}(x^3) + 15\frac{d}{dx}(x^2)$$

$$= 5\cdot 4x^{4-1} - 20\cdot 3x^{3-1} + 15\cdot 2x^{2-1} = 20x^3 - 60x^2 + 30x$$

$$\frac{d^3y}{dx^3} = 20\frac{d}{dx}(x^3) - 60\frac{d}{dx}(x^2) + 30\frac{d}{dx}(x)$$

$$= 20\cdot 3x^{3-1} - 60\cdot 2x^{2-1} + 30\cdot 1 = 60x^2 - 120x + 30$$

For maximum or minimum, $\frac{dy}{dx} = 0$

i.e. $5x^4 - 20x^3 + 15x^2 = 0$

$5x^2(x^2 - 4x + 3) = 0 \qquad 5x^2 = 0 \quad \therefore x = 0$

$x^2 - 3x - x + 3 = 0$

$x(x-3) - 1(x-3) = 0$

$(x-1)(x-3) = 0$

$(x-1) = 0 \qquad x - 3 = 0$

$\therefore x = 1 \qquad \therefore x = 3.$

$\therefore x = 0, 1, 3,$

When $x = 1$, $\frac{d^2y}{dx^2} = 20(1)^3 - 60(1)^2 + 30(1) = -10$ (negative).

$\therefore$ The function has maximum at $x = 1$ $\left[\because \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0\right]$

When $x = 3$, $\frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3) = 540 - 540 + 90 = 90$ (Positive)

$\therefore$ The function has minimum at $x = 3$ $\left[\because \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0\right]$

When $x = 0$, $\frac{d^2y}{dx^2} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ and

$\frac{d^3y}{dx^3} = 60(0)^2 - 120(0) + 30 = 30 \neq 0.$

$\therefore$ when $x = 0$, $\frac{d^2y}{dx^2} = 0.$

$\therefore$ At $x = 0$, the function attains neither maximum nor minimum and is a point of inflexion.

Illustration 6 : Show that the curve $y = x + \frac{1}{x}$ has one maxima and one minima.

Solution : $y = x + \frac{1}{x} = x + x^{-1}$

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) = 1 + (-1)\cdot x^{-1-1} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1) - \frac{d}{dx}(x^{-2}) = 0 - (-2)x^{-2-1} = 2x^{-3} = \frac{2}{x^3}$$

For maximum or minimum, $\frac{dy}{dx} = 0.$

$1 - \frac{1}{x^2} = 0$ (Cross multiplying).

$x^2 - 1 = 0.$

$(x+1)(x-1) = 0$ $\left[(a+b)(a-b) = a^2 - b^2\right]$

$x + 1 = 0 \qquad x - 1 = 0$

$\therefore x = -1 \qquad \therefore x = 1$

When $x = -1$, $\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = \frac{2}{-1} = -2$ (negative)

$\therefore$ The function has maximum value at $x = -1$ $\left[\because \frac{d^2y}{dx^2} < 0\right]$

When $x = 1$; $\frac{d^2y}{dx^2} = \frac{2}{(1)^3} = \frac{2}{1} = 2$ (Positive)

$\therefore$ The function has minimum value at $x = 1$.

$\therefore$ $x = 1$ is the only minimum point and $x = -1$ is the only maximum point on the real line.

Note : The function is not continuous at $x = 0$ and continuous elsewhere.

Illustration 7 : Find the maximum and minimum values of the function

$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x$$

Solution : Let $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x$

$$\frac{dy}{dx} = \frac{1}{3}\cdot\frac{d}{dx}(x^3) - \frac{1}{2}\cdot\frac{d}{dx}(x^2) - 6\cdot\frac{d}{dx}(x)$$

$$= \frac{1}{3}\cdot 3x^{3-1} - \frac{1}{2}\cdot 2x^{2-1} - 6\cdot 1 = x^2 - x - 6.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2) - \frac{d}{dx}(x) - \frac{d}{dx}(6) = 2\cdot x^{2-1} - 1 - 0 = 2x - 1.$$

For maximum or minimum, $\frac{dy}{dx} = 0$.

i.e.

$$x^2 - x - 6 = 0$$
$$x^2 - 3x + 2x - 6 = 0$$
$$x(x-3) + 2(x-3) = 0$$
$$(x+2)(x-3) = 0$$

$x + 2 = 0$ $\quad$ $x - 3 = 0$

$\therefore x = -2$ $\quad$ $\therefore x = 3$

When $x = -2$, $\frac{d^2y}{dx^2} = 2(-2) - 1 = -5$ (negative).

$\therefore$ The function has maximum at $x = -2$.

The maximum value $= \frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 - 6(-2) = \frac{22}{3}$.

When $x = 3$, $\frac{d^2y}{dx^2} = 2(3) - 1 = 5$ (positive)

$\therefore$ The function has minimum at $x = 3$.

The minimum value $= \frac{1}{3}(3)^3 - \frac{1}{2}(3)^2 - 6(3) = -\frac{27}{2}$.

Illustration 8 : State whether $y = \frac{1}{5}x^5 - \frac{13x^3}{3} + 36x - 9$ has a maximum or a minimum value. Find the values.

Solution : $y = \frac{1}{5}x^5 - \frac{13x^3}{3} + 36x - 9$.

$$\frac{dy}{dx} = \frac{1}{5}\cdot\frac{d}{dx}(x^5) - \frac{13}{3}\cdot\frac{d}{dx}(x^3) + 36\frac{d}{dx}(x) - \frac{d}{dx}(9).$$

$$= \frac{1}{5}\cdot 5x^{5-1} - \frac{13}{3}\cdot 3x^{3-1} + 36\cdot 1 - 0 = x^4 - 13x^2 + 36$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^4) - 13\frac{d}{dx}(x^2) + \frac{d}{dx}(36)$$

$$= 4\cdot x^{4-1} - 13\cdot 2x^{2-1} + 0 = 4x^3 - 26x$$

For maximum or minimum, $\frac{dy}{dx} = 0$

i.e.

$$x^4 - 13x^2 + 36 = 0$$
$$x^4 - 9x^2 - 4x^2 + 36 = 0$$
$$x^2(x^2 - 9) - 4(x^2 - 9) = 0$$
$$(x^2 - 4)(x^2 - 9) = 0$$

$(x^2 - 4) = 0$ $\quad$ $(x^2 - 9) = 0$

$x^2 = 4$ $\quad$ $x^2 = 9$

$\therefore x = \sqrt{4} = \pm 2.$ $\quad$ $\therefore x = \sqrt{9} = \pm 3$

$x = +2, -2, +3, -3.$

When $x = 2$, $\frac{d^2y}{dx^2} = 4(2)^3 - 26(2) = -20$ (negative).

$\therefore$ The function has maximum at $x = 2$.

The maximum value at $x = 2$ is $\frac{1}{5}(2)^5 - \frac{13(2)^3}{3} + 36(2) - 9 = \frac{521}{15}$.

When $x = -2$, $\frac{d^2y}{dx^2} = 4(-2)^3 - 26(-2) = 20$ (positive)

$\therefore$ The function attains minimum value at $x = -2$.

The minimum value $x = -2$, is $\frac{1}{5}(-2)^5 - \frac{13(-2)^3}{3} + 36(-2) - 9 = -\frac{791}{15}$

When $x = 3$, $\frac{d^2y}{dx^2} = 4(3)^3 - 26(3) = 30$ (positive)

$\therefore$ The function attains minimum value at $x = 3$.

The minimum value at $x = 3$ is $\frac{1}{5}(3)^5 - \frac{13(3)^3}{3} + 36(3) - 9 = \frac{153}{5}$.

When $x = -3$, $\frac{d^2y}{dx^2} = 4(-3)^3 - 26(-3) = -30$ (negative).

$\therefore$ The function has minimum at $x = -3$.

The maximum value at $x = -3$ is $\frac{1}{5}(-3)^5 - \frac{13(-3)^3}{3} + 36(-3) - 9 = -\frac{243}{5}$.

Illustration 9 : Find the extreme values of the function x^2e^{-x}.

Solution : Let $y = x^2e^{-x} = u \cdot v$

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^{-x}) \qquad \left[\frac{dy}{dx} = v \cdot \frac{d}{dx}u + u \cdot \frac{d}{dx}v\right]$$

$$= e^{-x} \cdot 2x + x^2(-e^{-x})$$

$$= 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\left[e^{-x}(2x - x^2)\right] = (2x - x^2)\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(2x - x^2)$$

$$= (2x - x^2)(-e^{-x}) + e^{-x}(2 - 2x)$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$e^{-x}(2x - x^2) = 0$$

$$(2x - x^2) = 0 \qquad e^{-x} = 0 \neq 0 \text{ for all } x$$

$$x(2 - x) = 0$$

$$x = 0 \quad 2 - x = 0 \quad \therefore x = 2$$

$$x = \infty, 0, 2$$

When $x = 0$, $\frac{d^2y}{dx^2} = \left(2(0) - (0)^2\right)\left(-e^{-0}\right) + e^{-0}\left(2 - 2(0)\right) = 2$ (positive)

$\therefore$ when $x = 0$, y attains its minimum.

The minimum value $= (0)^2e^{-0} = 0$.

When $x = 2$, $\frac{d^2y}{dx^2} = \left(2(2) - (2)^2\right)\left(-e^{-2}\right) + e^{-2}\left(2 - 2(2)\right)$

$$= (4 - 4)(-e^{-2}) + e^{-2}(-2) = -2e^{-2} \text{ (negative)}$$

$\therefore$ When $x = 2$, y attains its maximum.

$\therefore$ The maximum value $= 2^2 \cdot e^{-2} = 4e^{-2}$.

Illustration 10 : Find the maximum and minimum values of $f(x)$, where $f(x) = (x-1)^2(x+1)^3$ **(I.C.W.A.I. June 1974)**

Solution : Let $y = (x-1)^2(x+1)^3$

$$\frac{dy}{dx} = (x+1)^3\frac{d}{dx}(x-1)^2 + (x-1)^2\frac{d}{dx}(x+1)^3$$

$$= (x+1)^3 \cdot 2(x-1) + (x-1)^2 \cdot 3(x+1)^2$$

$$= (x+1)^2(x-1)[(x+1)2 + (x-1)3]$$

$$= (x+1)^2(x-1)[2x + 2 + 3x - 3]$$

$$= (x-1)(x+1)^2(5x-1)$$

$$\frac{d^2y}{dx^2} = (x-1)(x+1)^2 \frac{d}{dx}(5x-1) + (x-1)(5x-1)\frac{d}{dx}(x+1)^2 + (x+1)^2(5x-1)\frac{d}{dx}(x-1)$$

$$= (x-1)(x+1)^2 \cdot 5 + (x-1)(5x-1)\cdot 2(x+1) + (x+1)^2(5x-1)\cdot 1$$

For maxima or minima, $\frac{dy}{dx} = 0.$

$$(x-1)(x+1)^2(5x-1) = 0$$

$$\Rightarrow x = 1, \; x = -1, \; x = \frac{1}{5}.$$

When $x = 1$, $\frac{d^2y}{dx^2} = 0+0+4\cdot 4 = 16$ (positive)

$\therefore x = 1$ is a minimum point. Minimum value = 0.

When $x = -1$, $\frac{d^2y}{dx^2} = 0+0+0 = 0$

$\therefore x = -1$ is a point where there is neither minimum nor maximum. It is a point of inflexion.

When $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = \left(-\frac{1}{4}\right)\left(\frac{36}{25}\right)5+0+0 = -\frac{9}{5} < 0.$

$\therefore x = \frac{1}{5}$ is a maximum point.

$$\text{The maximum value} = \left(\frac{1}{5}-1\right)^2\left(\frac{1}{5}+1\right)^3 = \left(-\frac{4}{5}\right)^2\left(\frac{6}{5}\right)^3$$

$$= \frac{16}{25} \times \frac{216}{125} = \frac{3456}{3025}.$$

Illustration 11 : *For a firm under perfect competition, it is given that price of a car is Rs.3 lakhs and cost of production $C = 100 + 0.015x^2$. Find how many pleasure cars be produced to maximize the profit?. What would be the maximum profit?. What would be the profit when number of cars sold is 150?.*

Solution :

Given : Price per car = Rs. 3 lakhs; number of cars = x; Total cost = C;

Profit = Total Revenue − Total cost

i.e. $P = (3\cdot x) - (100 + 0\cdot 015x^2)$

$P = 3x - 100 - 0\cdot 015x^2.$

Profit is the function of number of cars produced.

$$\therefore \frac{dp}{dx} = \frac{d}{dx}(3x - 100 - 0\cdot 015x^2)$$

$$= 3\cdot\frac{d}{dx}(x) - \frac{d}{dx}(100) - 0\cdot 015\frac{d}{dx}(x^2)$$

$$= 3\cdot 1 - 0 - 0\cdot 015 \times 2x^{2-1} = 3 - 0.03x$$

$$\frac{d^2p}{dx^2} = \frac{d}{dx}(3 - 0\cdot 03x) = \frac{d}{dx}(3) - 0\cdot 03\frac{d}{dx}(x) = -0.03 \text{ (negative)}.$$

For maximum or minimum, $\frac{dp}{dx} = 0$

$$3 - 0.03x = 0$$

$$-0.03x = -3.$$

$$\therefore x = \frac{-3}{-0.03} = 100.$$

When $x = 100$; $\frac{d^2p}{dx^2} = -0.03 < 0.$

Therefore it is concluded that profit will be maximum when 100 pleasure cars are produced.

$\therefore$ Maximum profit $= 3(100) - 100 - 0.015(100)^2$

= Rs. 50 lakhs.

When $x = 150$; The profit $= 3(150) - 100 - 0.015(150)^2$ = Rs. 12.5 lakhs.

Illustration 12 : A producer can sell x units per month at a price of Rs. 300 − 2 x each. The cost of production worked at 20 x + 1000. Find the number of units to be produced so that the profit will be the maximum. Also calculate the expected maximum profit.

Solution : Given : Price of a car = 300 − 2 x ; C = 20 x + 1000; number of cars produced = x.

Total cost = 20 x + 1000.

Profit = Total Revenue − Total cost

i.e. $P = x\cdot(300-2x)-(20x+1000)$

$P = 300x - 2x^2 - 20x - 1000 = 280x - 2x^2 - 1000.$

$$\frac{dp}{dx} = 280\cdot\frac{d}{dx}(x) - 2\cdot\frac{d}{dx}(x^2) - \frac{d}{dx}(1000) = 280 - 4x.$$

$$\frac{d^2p}{dx^2} = \frac{d}{dx}(280) - 4\frac{d}{dx}(x) = 0 - 4.1 = -4 \text{ (negative).}$$

For maximum or minimum, $\frac{dp}{dx} = 0$

$280 - 4x = 0$

$4x = 280$

$\therefore x = 280/4 = 70.$

When $x = 70$, $\frac{d^2p}{dx^2} < 0$.

∴ The profit is maximum when the number of units produced is 70.

∴ The maximum amount of profit $= 280(70) - 2(70)^2 - 1000$ = Rs. 8800.

Illustration 13 : The cost function for producing x units of a product is $C = x^3 - 12x^2 + 48x + 11$ (in rupees) and the revenue function is $R = 83x - 4x^2 - 21$. Find the output for which profit is maximum.

(I.C.W.A.I. June 1986).

Solution : Given : $R = 83x - 4x^2 - 21$; $C = x^3 - 12x^2 + 48x + 11$

Profit, $P = R - C$

i.e. $P = 83x - 4x^2 - 21 - (x^3 - 12x^2 + 48x + 11)$

$P = 83x - 4x^2 - 21 - x^3 + 12x^2 - 48x - 11$

$P = -x^3 + 8x^2 + 35x - 32.$

$$\frac{dP}{dx} = -\frac{d}{dx}(x^3) + 8\cdot\frac{d}{dx}(x^2) + 35\cdot\frac{d}{dx}(x) - \frac{d}{dx}(32)$$

$= -3x^2 + 16x + 35.$

$$\frac{d^2P}{dx^2} = -3\frac{d}{dx}(x^2) + 16\frac{d}{dx}(x) + \frac{d}{dx}(35)$$

$= -6x + 16.$

For maxima or minima, $\frac{dP}{dx} = 0.$

$-3x^2 + 16x + 35 = 0.$

$\Rightarrow 3x^2 - 16x - 35 = 0$

$3x^2 - 21x + 5x - 35 = 0.$

$3x(x-7) + 5(x-7) = 0$

$(3x+5)(x-7) = 0$

$3x + 5 = 0.$ $x - 7 = 0.$

$3x = -5$ $\therefore x = 7.$

$\therefore x = -\frac{5}{3}.$

Since $x = -\frac{5}{3}$ is not admissible, production quantity should be positive

$\therefore x = 7.$

When $x = 7$, $\frac{d^2p}{dx^2} = -6(7) + 16 = -26$ (negative).

∴ Maximum profit is achieved when the output is 7 units.

∴ Maximum profit $= (-7)^3 + 8(7)^2 + 35(7) - 32$ = Rs. 262.

Illustration 14: A certain manufacturing concern has the total cost function as $C = 15 + 9x - 6x^2 + x^3$. Find when the total cost is minimum.

Solution : Given $C = 15 + 9x - 6x^2 + x^3$.

where x denote no. of units produced.

Cost is the non linear function of number of units produced. That is the rate of change in cost is the function of rate of change in units produced.

$$\therefore \frac{dc}{dx} = \frac{d}{dx}(15) + 9 \cdot \frac{d}{dx}(x) - 6 \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$$

$$= 9 - 12x + 3x^2.$$

$$\frac{d^2c}{dx^2} = \frac{d}{dx}(9) - 12 \cdot \frac{d}{dx}(x) + 3 \cdot \frac{d}{dx}(x^2)$$

$$= -12 + 6x.$$

For maximum or minimum, $\frac{dc}{dx} = 0$.

$$3x^2 - 12x + 9 = 0.$$

$$3(x^2 - 4x + 3) = 0.$$

$$x^2 - 3x - x + 3 = 0.$$

$$x(x-3) - 1(x-3) = 0$$

$$(x-1)(x-3) = 0$$

$x - 1 = 0$ $\quad$ $x - 3 = 0$

$\therefore x = 1$ $\quad$ $\therefore x = 3.$

When $x = 1$, $\frac{d^2c}{dx^2} = 6(1) - 12 = -6$ (negative)

$\therefore$ Cost is maximum when $x = 1$.

When $x = 3$, $\frac{d^2c}{dx^2} = 6(3) - 12 = 6$ (positive)

$\therefore$ Cost is minimum when $x = 3$.

$\therefore$ The optimal production quantity is 3 units.

Illustration 15 : The total profit y in rupees of a drug company from the manufacture and sale of x drug bottles is given by $y = -\frac{x^2}{400} + 2x - 80$.

(i) How many drug bottles must the company sell to achieve the maximum profit?

(ii) What is the profit per drug bottle when this maximum is achieved?

(C.A., Nov. 1985).

Solution : Given, $y = -\frac{x^2}{400} + 2x - 80.$

The rate of change in profit is the function of rate of change in number of drug bottles manufactured and sold.

$$\therefore \frac{dy}{dx} = -\frac{1}{400} \cdot \frac{d}{dx}(x^2) + 2 \cdot \frac{d}{dx}(x) - \frac{d}{dx}(80)$$

$$= -\frac{1}{400} \cdot 2x^{2-1} + 2 \cdot 1 - 0 = -\frac{x}{200} + 2.$$

$$\frac{d^2y}{dx^2} = -\frac{1}{200} \cdot \frac{d}{dx}(x) + \frac{d}{dx}(2) = -\frac{1}{200}. \text{ (negative)}$$

For maximum or minima, $\frac{dy}{dx} = 0$.

$$\frac{-x}{200} + 2 = 0. \Rightarrow \frac{-x + 400}{200} = 0. \Rightarrow \therefore x = 400.$$

When $x = 400$, $\frac{d^2y}{dx^2} = -\frac{1}{200} < 0$.

$\therefore$ The company get maximum profit when sale is 400 bottles.

$\therefore$ The maximum profit $= \frac{-(400)^2}{400} + 2(400) - 80$ = Rs. 1120.

(ii) Profit per drug bottle $= \frac{1120}{400}$ = Rs. 2.80

Illustration 16 : Under perfect competition, the price of a toy has been fixed at Rs.6. The total cost function of a firm is given by $C = \frac{1}{3}q^3 - 5q^2 + 15q + 10$. Find the number of toys produced at which profit will be maximum. Also calculate the amount of maximum profit.

Solution : Given : Price of a toy = Rs. 6; no. of toys produced = q.

Profit = Total Revenue − Total Cost.

$$P = 6 \cdot q - \left(\frac{1}{3}q^3 - 5q^2 + 15q + 10\right)$$

$$P = 6q - \frac{1}{3}q^3 + 5q^2 - 15q - 10 = -\frac{1}{3}q^3 + 5q^2 - 9q - 10$$

$$\frac{dP}{dq} = -\frac{1}{3} \cdot \frac{d}{dq}(q^3) + 5 \cdot \frac{d}{dq}(q^2) - 9 \cdot \frac{d}{dq}(q) - \frac{d}{dq}(10)$$

$$= -\frac{1}{3} \cdot 3q^{3-1} + 5 \cdot 2q^{2-1} - 9 \cdot 1 - 0 = -q^2 + 10q - 9.$$

$$\frac{d^2P}{dq^2} = -\frac{d}{dq}(q^2) + 10 \cdot \frac{d}{dq}(q) - \frac{d}{dq}(9) = -2q + 10.$$

For maxima or minima, $\frac{dP}{dq} = 0.$

$$-q^2 + 10q - 9 = 0$$

$$q^2 - 10q + 9 = 0. \quad \Rightarrow \quad q^2 - 9q - 1q + 9 = 0.$$

$$\Rightarrow \quad q(q-9) - 1(q-9) = 0.$$

$$(q-1)(q-9) = 0$$

$$q - 1 = 0 \qquad q - 9 = 0$$

$$\therefore q = 1 \qquad \therefore q = 9$$

When $q = 1$, $\frac{d^2P}{dq^2} = -2(1) + 10 = 8$ (positive).

∴ The profit is minimum when the output is 1.

When $q = 9$, $\frac{d^2P}{dq^2} = -2(9) + 10 = -8$ (negative).

∴ The profit is maximum when the output is 9 units.

∴ The maximum profit $= -\frac{1}{3}(9)^3 + 5(9)^2 - 9(9) - 10 =$ Rs. 71.

Marginal cost : Marginal cost is the unit change in cost because of additional unit change in volume. In other words, marginal cost denotes the rate of change in cost with respect to change in number of units produced. Hence **first derivative of a cost function is nothing but the function of marginal cost.**

For example, if $y = 4x^3 - 3x^2 + 7$.

where y is cost and x is the quantity (i.e. cost is a function of quantity), then,

marginal cost function $= \frac{dy}{dx} = \frac{d}{dx}(4x^3 - 3x^2 + 7) = 12x^2 - 6x.$

Average cost : The cost function per unit is called average cost. If C (x) is the total cost, then $\frac{C(x)}{x}$ is the average cost.

Illustration 17 : The cost function of a firm is $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$ where x stands for output and C for cost. Calculate

(i) the output at which marginal cost is minimum.

(ii) the output at which average cost is minimum.

(C.A. Nov., 1987)

Solution : $c = 300x - 10x^2 + \frac{1}{3}x^3$.

(i) Marginal Cost function (mc)

$$= \frac{dC}{dx} = 300 \cdot \frac{d}{dx}(x) - 10 \cdot \frac{d}{dx}(x^2) + \frac{1}{3} \cdot \frac{d}{dx}(x^3)$$

$$= 300 \cdot 1 - 10 \cdot 2x^{2-1} + \frac{1}{3} \cdot 3x^{3-1} = 300 - 20x + x^2.$$

Differentiating mc $= \frac{d}{dx}(mc)$

$$= \frac{d}{dx}(300) - 20 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(x^2)$$

$$= -20 + 2x.$$

$$\frac{d^2mc}{dx^2} = \frac{d}{dx}(-20 + 2x) = -\frac{d}{dx}(20) + 2\frac{d}{dx}(x) = 2 \text{ (positive)}$$

For maximum or minimum, $\frac{dmc}{dx} = 0.$

$$-20 + 2x = 0$$

$$2x = 20$$

$$\therefore x = 10.$$

When $x = 10$, $\frac{d^2mc}{dx^2} = 2 > 0.$

$\therefore$ Marginal cost is minimum when the output is 10 units.

(ii) Average cost $\bar{y} = \frac{c(x)}{x} = \frac{300x}{x} - \frac{10x^2}{x} + \frac{1 \cdot x^3}{3x}$

$$= 300 - 10x + \frac{x^2}{3}.$$

$$\frac{d}{dx}(\bar{y}) = \frac{d}{dx}(300) - 10 \cdot \frac{d}{dx}(x) + \frac{1}{3} \cdot \frac{d}{dx}(x^2)$$

$$= -10 + \frac{2}{3}x.$$

$$\frac{d^2\bar{y}}{dx^2} = -\frac{d}{dx}(10) + \frac{2}{3} \cdot \frac{d}{dx}(x) = \frac{2}{3} \text{ (positive)}.$$

For maximum or minimum $\frac{d\bar{y}}{dx} = 0.$

$$-10 + \frac{2}{3}x = 0.$$

$$\frac{2}{3}x = 10$$

$$\therefore x = 10 \cdot \frac{3}{2} = 15.$$

When $x = 15$, $\frac{d^2\bar{y}}{dx^2} = \frac{2}{3} > 0.$

$\therefore$ The average cost is minimum when the output is 15.

Illustration 18: The cost function of Raman & Co is $c = x^3 - 3x^2 + 3x$ where c represents cost and x represents quantity. Find

(i) the equation for marginal cost function

(ii) the output at which marginal cost is minimum and

(iii) the output at which average cost is minimum.

Solution : $c = x^3 - 3x^2 + 3x.$

(i) Marginal Cost function (mc)

$$\frac{dc}{dx} = \frac{d}{dx}(x^3) - 3 \cdot \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(x)$$

$$= 3x^{3-1} - 3 \cdot 2x^{2-1} + 3 \cdot 1 = 3x^2 - 6x + 3.$$

(ii) Differentiating mc $= \frac{d}{dx}(mc)$

$$= 3 \cdot \frac{d}{dx}(x^2) - 6 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(3)$$

$$= 3 \cdot 2x^{2-1} - 6 \cdot 1 = 6x - 6.$$

$$\frac{d^2mc}{dx^2} = 6\cdot\frac{d}{dx}(x) - \frac{d}{dx}(6) = 6 \text{ (positive)}$$

For maxima or minima, $\frac{dmc}{dx} = 0.$

$$6x - 6 = 0$$
$$6x = 6$$
$$\therefore x = 1.$$

When $x = 1$, $\frac{d^2mc}{dx^2} = 6 > 0.$

$\therefore$ The marginal cost is minimum when the output is 1.

(iii) Average cost $\bar{y} = \frac{c}{x} = \frac{x^3 - 3x^2 + 3x}{x} = x^2 - 3x + 3.$

$$\frac{d\bar{y}}{dx} = \frac{d}{dx}(x^2) - 3\cdot\frac{d}{dx}(x) + \frac{d}{dx}(3) = 2x - 3.$$

$$\frac{d^2\bar{y}}{dx} = 2\cdot\frac{d}{dx}(x) - \frac{d}{dx}(3) = 2. \text{ (positive).}$$

By equating $\frac{d\bar{y}}{dx} = 0.$

$$2x - 3 = 0$$
$$2x = 3$$
$$\therefore x = 3/2$$

When $x = \frac{3}{2}$, $\frac{d^2\bar{y}}{dx^2} = 2 > 0.$

$\therefore$ **Average cost is minimum when the output is 1.5 units.**

Illustration 19 : A Sitar manufacturer notices that he can sell x sitars per week at 'p' rupees each where $5x = 375 - 3p$. The cost of production is $\left(500 + 13x + \frac{1}{5}x^2\right)$ **rupees.** Show that maximum profit is achieved when the production and sale volume is 30 **sitars** per week.

Solution : Given $5x = 375 - 3p$

$$\therefore p = \frac{375 - 5x}{3}$$

Hence total revenue $= x\cdot p = x\left(\frac{375 - 5x}{3}\right) = 125x - \frac{5}{3}x^2.$

Profit = Total Revenue - Total Cost

$$P = 125x - \frac{5}{3}x^2 - (500 + 13x + \frac{1}{5}x^2)$$

$$P = 125x - \frac{5}{3}x^2 - 500 - 13x - \frac{1}{5}x^2$$

$$P = 112x - \frac{28}{15}x^2 - 500.$$

$$\frac{dp}{dx} = 112\cdot\frac{d}{dx}(x) - \frac{28}{15}\cdot\frac{d}{dx}(x^2) - \frac{d}{dx}(500) = 112 - \frac{56}{15}x$$

$$\frac{d^2p}{dx^2} = \frac{d}{dx}(112) - \frac{56}{15}\cdot\frac{d}{dx}(x) = -\frac{56}{15} \text{ (negative).}$$

For maximum or minimum, $\frac{dp}{dx} = 0.$

$$112 - \frac{56x}{15} = 0.$$

$$\frac{1680 - 56x}{15} = 0. \Rightarrow 56x = 1680. \quad \therefore x = 30.$$

When $x = 30$, $\frac{d^2p}{dx^2} = -\frac{56}{15} < 0$.

$\therefore$ The company's profit is maximum when 30 sitars were produced.

Illustration 20 : The cost C of manufacturing a certain article is given by the formula $C = 5 + \frac{48}{x} + 3x^2$ where x is the number of articles manufactured. Find the minimum value of C for optimal production quantity.

Solution : Given $C = 5 + \frac{48}{x} + 3x^2$

$$\frac{dC}{dx} = \frac{d}{dx}(5) + 48 \cdot \frac{d}{dx}(x^{-1}) + 3 \cdot \frac{d}{dx}(x^2) = -48 \cdot x^{-2} + 6x.$$

$$\frac{d^2C}{dx^2} = -48 \cdot \frac{d}{dx}(x^{-2}) + 6 \cdot \frac{d}{dx}(x) = -48(-2)x^{-2-1} + 6 \cdot 1 = 96x^{-3} + 6.$$

For maximum or minimum, $\frac{dc}{dx} = 0$.

i.e. $\frac{-48}{x^2} + 6x = 0$.

$\Rightarrow 6(x^3 - 8) = 0$.

$\Rightarrow x^3 = 8 = (2)^3$.

$\therefore x = 2$.

When $x = 2$, $\frac{d^2c}{dx^2} = \frac{96}{(2)^3} + 6 = \frac{96}{8} + 6 = 18$. (positive)

$\therefore$ Cost is minimum when the production quantity is 2.

Minimum value of $C = 5 + \frac{48}{2} + 3(2)^2 = $ Rs. 41.

Illustration 21: If the demand function of a monopolist is $3q = 98 - 4p$ and the average cost is $3q + 2$ where q is the out put and p is the price, find the maximum profit of the monopolist.

Solution : Total cost = average cost x output

$= (3q + 2)q = 3q^2 + 2q$.

Given $3q = 98 - 4p$

$4p = 98 - 3q$

$p = \frac{98 - 3q}{4}$

Selling price $= qp = q\left(\frac{98-3q}{4}\right) = \frac{49}{2}q - \frac{3}{4}q^2$

Profit = Selling price - Cost price

$P = \frac{49}{2}q - \frac{3}{4}q^2 - (3q^2 + 2q)$

$= \frac{45}{2}q - \frac{15}{4}q^2$

$\frac{dP}{dq} = \frac{45}{2} - \frac{15}{2}q$, $\frac{d^2P}{dq^2} = -\frac{15}{2} < 0$.

For maximum or minimum,

$\frac{dP}{dq} = 0 \Rightarrow \frac{45}{2} - \frac{15}{2}q = 0. \Rightarrow 15q = 45$

$\therefore q = 3$.

Since $\frac{d^2P}{dq^2} = -\frac{15}{2} < 0$, at $q = 3$, profit is maximum.

Maximum Profit $= \frac{45}{2} \cdot 3 - \frac{15}{4}(3)^2 = $ Rs. 33.75.

COMMERCIAL ARITHMETIC

4.1. SIMPLE INTEREST

4.1.1. Introduction

Organisations and individuals borrow money for their use from others. **Interest is the extra money paid by the organisation for having used the money lent by others. The money received for use is called Principal.** The interest is usually paid at the end of specified equal intervals of time; i.e. annually, half-yearly, quarterly or monthly. **The sum total of the principal and interest is called Amount.**

The rate of interest is the ratio of the interest charged in one unit of time to the Principal. Generally the unit of time is one year. The calculation of interest is divided into two i.e. simple interest and compound interest.

4.1.2. Simple Interest :

When the interest is calculated on principal at a uniform rate it is called simple interest. The important feature of the simple interest is that the interest whether paid or not, does not get accumulated along with the principal for the calculation of further interest. For example assume a sum of Rs. 20, 000 is borrowed at 10%. At the end of the 1st year, the simple interest works at Rs. 2000. Even if it is not actually paid, the next year's interest will be calculated only on Rs. 20,000 which is the principal. Simple interest will be calculated as follows :

$$SI = \frac{PNR}{100} \Rightarrow P = \frac{100 \times SI}{NR}$$

where SI = Simple Interest, P = Principal; N = Period (no. of years), R = Rate of interest.

(or)

$$SI = PNi$$

$$\text{where } i = \frac{R}{100}$$

The other formulae are :

Principal : $P = \frac{SI}{Ni}$

Period : $N = \frac{SI}{Pi}$

Rate of interest : $R = \frac{SI}{PN} \times 100$

Amount : $A = P(1 + Ni) \Rightarrow A = P\left(1 + \frac{NR}{100}\right)$

Note : **Generally, the period 'N' is expressed in terms of years. Suppose if the period is given in term of days (or) months, it should be converted into years**

Illustration 1 : Calculate the total interest on Rs. 500 for 73 days, Rs. 720 for 14 weeks and on Rs.900 for 3 months at 6% per annum.

Solution : Simple interest = PNi $\left[\frac{PNR}{100}\right]$

Interest on Rs. 500 for 73 days $= 500 \times \frac{73}{365} \times 0.06 =$ Rs. 6

Interest on Rs. 720 for 14 weeks $= 720 \times \frac{14}{52} \times 0.06 =$ Rs.11.63

Interest on Rs. 900 for 3 months $= 900 \times \frac{3}{12} \times 0.06 =$ Rs.13.50

∴ Total interest = Rs. 6 + Rs. 11.63 + Rs. 13.50 = Rs. 31.13.

Illustration 2 : Find the principal sum which yields simple interest of Rs. 77 in 8 years at 3 ½ % per annum.

Solution : Given SI = Rs. 77; N = 8; $i = \frac{3.5}{100}$

$$P = \frac{SI}{Ni} = \frac{77}{8 \times 0.035} = \text{Rs. } 275 \quad = \frac{100 \times 77}{8 \times 3.5} = \frac{770}{28} = \text{Rs } 275$$

∴ The principal sum = Rs. 275.

Illustration 3 : What sum of money invested at 10% simple interest will amount to Rs. 1200 in 2 years ?

Solution : Given A = Rs. 1200; N = 2; $i = \frac{10}{100} = 0.1$

$$A = P(1 + Ni)$$
$$1200 = P(1 + 2 \times 0.1)$$
$$1200 = 1.2\,P$$
$$\therefore P = \frac{1200}{1.2} = \text{Rs. } 1000$$

∴ The Principal sum invested = Rs. 1000.

Illustration 4 : Find out the period by which a sum of Rs. 1200 will amount to Rs. 1700 at 12% p.a.

Solution : Given P = Rs.1200; A = Rs. 1700; $i = \frac{12}{100} = 0.12$

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$$SI = A - P$$
$$= \text{Rs. } 1700 - \text{Rs. } 1200 = \text{Rs. } 500.$$
$$N = \frac{SI}{Pi} = \frac{500}{1200 \times 0.12} = 3.47 \text{ years.}$$

Illustration 5 : Find the rate of interest per annum if the simple interest on a principal of Rs. 5000, is Rs. 800 for 4 years.

Solution : Given P = Rs. 5000; SI = Rs. 800; N = 4.

$$R = \frac{SI}{PN} \times 100$$
$$= \frac{800}{5000 \times 4} \times 100 = 4\% \text{ p.a}$$

∴Rate of interest = 4% p.a.

Illustration 6 : In what time will a sum of money double itself at 10% p.a. simple interest?

Solution : Let P = 100, then A = 200, i = 0.1; SI = A – P = 200 – 100 = 100

$$N = \frac{SI}{Pi} = \frac{100}{100 \times 0.1} = 10$$

∴A sum of money will double itself at 10% p.a. in 10 years.

Illustration 7 : Find the number of years in which a sum of money will triple itself at 10% p.a. simple interest.

Solution : Let P = Rs. 100, then A = Rs. 300, i = 0.1;

$$SI = A - P = \text{Rs. } 300 - \text{Rs. } 100 = \text{Rs.}200$$
$$N = \frac{SI}{Pi} = \frac{200}{100 \times 0.1} = 20$$

∴A sum of money will triple itself at 10% p.a. in 20 years.

Illustration 8 : On October 1st 2002, a loan of Rs.365 was repaid with simple interest 7% p.a. by a cheque of Rs. 373.40. On what date was the money borrowed.

Solution : Given P = Rs. 365, A = Rs.373.40, i = 7/100 = 0.07

Simple interest = A – P

= Rs. 373.40 – Rs. 365 = Rs. 8.40

$$N = \frac{SI}{Pi} = \frac{8.40}{365 \times 0.07} \times 365 = 120 \text{ days.}$$

(year was converted into days)

Date of Repayment = 1^{st} October

No. of days amount borrowed = 120 days.

∴ Date of borrowing = 1^{st} Oct. 2002 – 120 days.

= 1^{st} Oct. – { (Oct.) 1 + (Sep.) 30 + (Aug.) 31 + (July) 31 + June 27 }

= 3^{rd} June 2002.

Illustration 9 : An employee deposits his salary of Rs.7500 p.m. into a post office savings account at the beginning of every month at 8% p.a. simple interest. He withdraws Rs.4500 p.m. during each month. Find out the balance standing to his credit at the end of 2 years.

Solution :

Savings at the end of every month = 7500 – 4500 = Rs. 3000

Let the monthly savings be Rs. P

∴ P = Rs. 3000; i = 8/100 = 0.08;

N = (23 + 22 + ... + 0) months

$\therefore N = \frac{n}{2}(a + l)$, where a - first term, l - last term and n - number of terms.

$N = \frac{24}{2} \times (23 + 0)$ = 276 months or 23 years.

∴ Simple interest = 3000 x 23 x 0.08 = Rs. 5520.

∴ Amount standing at his credit at the end of 3 years

= Rs. (3000 x 24) + 5520 = 72000 + 5520.

= Rs. 77,520.

Illustration 10 : A man deposits a certain sum of money into a bank. It amounts to Rs.12,325 in 8 years and amounts to Rs.13, 565 in 10 years. Find the sum invested and also the rate of simple interest offered by the bank.

Solution :

Amount at the end of 10^{th} year = Rs. 13,565

Amount at the end of 8^{th} year = Rs. 12,325

Simple interest for 2 years = 13,565 – 12,325 = Rs. 1240

Simple interest for one year $= \frac{1240}{2}$ = Rs. 620

S.I. for 8 years = 8 x 620 = Rs. 4960

Sum Invested = Amount – Simple interest

= Rs. 12,325 – Rs. 4960 = Rs. 7365.

P = Rs. 7365; SI = Rs. 620; N = 1 year

Rate of interest $= \frac{SI}{PN} \times 100$

$= \frac{620 \times 100}{7365 \times 1} = 8.42\%$

∴ The man invested Rs. 7365 at 8.42 % p.a. simple interest.

Illustration 11 : A man wants to deposit a sum of Rs. 4,55, 000 in the name of his two sons aged 12 and 15 years in such a manner that the two sons get the same amount when they attain the age of 18 years. Assuming the rate of simple interest to be 11% p.a. find out how much should be deposited in their name in the beginning?

Solution :

Let the sum deposited in the name of elder son be **x**

Share of the younger son = 4,55,000 – **x**

The elder son has 3 years and the younger son has 6 years to attain the age of 18 years.

Elder son : P = **x**; n = 3 years; i = 11/100 = 0.11 ;

ni = 3 x 0.11 = 0.33.

Amount available at his 18 years of age. = P (1 + n i)

= **x** (1 + 0.33)

= 1.33 **x** .... (1)

Younger son : P = (4,55,000 - **x**); n = 6; i = 0.11; ni = 6 x 0.11 = 0.66

Amount available at his age of 18 years = P (1 + n i)

= (4,55,000 – **x**) (1 + 0.66)

= (4,55,000 – **x**) (1.66)

= 7,55,300 – 1.66 **x** ..... **(2)**

From the problem, the total amount in both cases are equal.

i.e. 1.33 **x** = 7,55,300 – 1.66**x**

1.33 **x** + 1.66 **x** = 7,55,300

2.99 **x** = 7,55,300

$x = \frac{7,55,300}{2.99}$ = Rs. 2,52,609.

∴ The amount deposited in the name of elder son = Rs. 2,52,609

The amount deposited to the name of the younger son

= Rs. 4,55,000 – Rs. 2,52,609 = Rs. 2,02,391.

Illustration 12 : Mr. X lent Rs.7200 at simple interest partly at 6% p.a. and partly at 7% p.a. If the interest received after one year is Rs. 450 how much did he lend at different rates of interest ?.

Solution :

Let the amount lent at 6% p.a. be x.

$\therefore$ Amount lent at 7% p.a. = (7200 – x)

SI on x for one year at 6% p.a. = P n i = x × 1 × 0.06 = 0.06x

SI on (7200 - x) for one year at 7% p.a.= P n i

$= (7200 - x) \times 1 \times 0.07 = 504 - 0.07x$

Given total interest received is Rs. 450,

$$\text{i.e. } 0.06x + 504 - 0.07x = 450.$$

$$0.01x = 54$$

$$\therefore x = \frac{54}{0.01} = \text{Rs. } 5400$$

$\therefore$ Amount lent at 6% p.a. = Rs. 5400

$\therefore$ Amount lent at 7% p.a. = Rs. 7200 – Rs. 5400 = Rs. 1800

Illustration 13 : A certain amount of money was invested at 12% p.a. simple interest and after 6 months an equal amount was invested at 14% p.a. simple interest. Find the period in which the amount in each case becomes Rs.6500. Also find out how much money was invested in each case.

Solution :

Let amount invested be Rs. P for a period n.

Given A = 6500; i = 0.12 in the first case

$$\text{Then } A = P(1 + n\,i)$$

$$\text{i.e. } 6500 = P(1 + 0.12n) \quad \text{.......... (1)}$$

Given A = 6500; i = 0.14; n = (n – 0.5) in the second case.

$$\therefore 6500 = P[1 + 0.14(n - 0.5)] \quad \text{.......... (2)}$$

Dividing (1) by (2)

$$\frac{6500}{6500} = \frac{P[1 + 0.12n]}{P[1 + 0.14n - 0.07]}$$

$$1 = \frac{1 + 0.12n}{1 + 0.14n - 0.07}$$

$$1 + 0.12n = 1 + 0.14n - 0.07$$

$$0.07 = 0.14n - 0.12n = 0.02n$$

$$\therefore n = \frac{0.07}{0.02} = 3.5 \text{ years}$$

∴ The no. of years in which the amount becomes Rs. 6500 = 3.5 years.

Put the value of 'n' in equation (1), we get

$$6500 = P(1 + 3.5 \times 0.12)$$

$$6500 = P(1 + 0.42) = 1.42\,P$$

$$\therefore P = \frac{6500}{1.42} = \text{Rs. } 4577.46$$

Thus the sum invested in each case = Rs. 4577.46.

University Examination Questions & Answers

Illustration 14 : What is the simple interest earned by Rs.6000 at 15% p.a. in 2 years?

[B.Com., M.K.U. Nov.'94]

Solution : Given P = Rs. 6000; N = 2; R = 15%; i = 0.15

Simple interest = PNI

= 6000 x 2 x 0.15 = Rs. 1800

Illustration 15 : Rs. 80 is deposited on the 1st day of every month for 3 years in savings account. Find the final sum if the 5% simple interest is offered by the Bank.

[B.Com., M.K.U. April '94]

Solution : Let the monthly deposit be Rs. 'P'

P = 80; i = 0.05

N = 36 months (for the first instalment of Rs. 80) + 35 months (for the second instalment of Rs.80) + ... + 1 month (for the last instalment of Rs. 80).

$$\therefore N = \frac{n}{2}(a + l)\text{, where a - first term, l - last term and n - number of terms.}$$

666/12 = 55.5 years

$$= \frac{36}{2} \times (36 + 1) = 18 \times 37 = 666 \text{ months or } 55.5 \text{ years.}$$

∴ Simple interest = PNI = 80 x 55.5 x 0.05 = Rs. 222

∴ Final sum at the end of 3 years = (80 x 36) + 222

= 2880 + 222 = Rs. 3102.

Illustration 16 : If Rs. 450 amounts to Rs.504 in 3 years at simple interest, what will Rs.650 amount to in 2 years 6 months, the rate of interest being the same in both cases. [B.Com., M.K.U., Nov., 2003]

Solution : Given : P = Rs. 450; A = Rs. 504; N = 3; R = ?

Simple Interest = A − P

= 504 − 450 = Rs. 54.

$$\text{Rate of Interest : } R = \frac{SI}{PN} \times 100$$

$$= \frac{54}{450 \times 3} \times 100 = \frac{54}{1350} \times 100 = 4\% \text{ p.a.}$$

Again, Given : P = 650; N = 2.5; R = 4% i.e. i = 0.04

SI = PNi = 650 x 2.5 x 0.04 = Rs. 65.

∴ Amount = P + SI

= 650 + 65 = Rs. 715.

Illustration 17 : What Principal will amount to Rs. 2250 in $3\frac{1}{3}$ years at $2\frac{1}{7}$ % simple interest ?. [B.Com., M.K.U., April, 2006]

Solution : Given : A = Rs. 2250; N = 3.33; R = 2.143% i.e. I = 0.0214

We know,

$$A = P(1 + Ni)$$

$$2250 = P(1 + 3.33 \times 0.0214)$$

$$2250 = 1.071\,P$$

$$P = \frac{2250}{1.071} = \text{Rs. } 2100.$$

EXERCISE

I Choose the correct answer :

(1) The simple interest on Rs. 50,000 at 10 % p.a. for 10 years is
(a) 25, 000 (b) 30,000 (c) 5 0,000 (d) 5000

(2) Simple interest on 20,000 at 2% p.a. for one year is
(a) 4,800 (b) 4,000 (c) 800 (d) 400

(3) The total interest on Rs. 12,000 for 9 months at the rate of 9% p.a. simple interest is
(a) 810 (b) 972 (c) 900 (d) none

ANSWERS

I Choose the correct answer : (1) c (2) d (3) a (4) b (5) c (6) b (7) a (8) c (9) b (10) d (11) a (12) b (13) b (14) a (15) d

II Answer the following : (1) 30,000 (2) 660; 5060 (3) (a) 420 ; 3420 (b) 35.20; 1495.20 (c) 1250; 26250 (d) 1250; 51,250 (e) 29.25; 3279.25 (4). 4709.25 (5) 1000 (6) 4800 (7). 500 (8) 6.67% (9). 10% (10). 3% (11). 1 (12). 2 (13). 11/2 (14) 4 (15) 12.5 (16) 77.63; 1877.63 (17) 2nd Ap. 2002 (18) 10,040 (19) 1,20,500; 6.47% (20). 600; 6.5% (21) 80,000, 50,000 (22) 400, 1200 (23) 900; 600 (24). 3.75,2000.

4.2. COMPOUND INTEREST

The compound interest is calculated recursively on the amount accumulated (i.e. principal plus interest) at the end of each year. The interest due at the end of the first year, if unpaid, is added to the original principal and the added amount is taken as the principal for the second year and thus this process being continued for the given period. **In other words interest is also given for the interest of the previous unit period.** In such case, we say that the interest has been compounded. The compound interest varies from period, as the principal vary.

The principle of compounding is also used to solve allied problems like growth of population, depreciation of assets etc. Thus the sum by which the original principal is increased at the end of all the specified conversion period is called **Compound interest** for the given period. Generally the **conversion period is one year.** If the **conversion period is half-year then the interest is compounded half-yearly.**

Formula for compound interest :

$$A = P\left(1 + \frac{R}{100}\right)^n$$

where, A = Amount at the end of 'n' years.
P = Principal
r = rate of interest per annum
n = the period in years

(or)

$$A = P(1 + i)^n \quad \text{where } i = R/100$$

Common logarithms are very useful to solve problems on compound interest. The formula for compound interest admits logarithmic computation.

We can determine either amount or principal or the period or rate of interest by making use of the logarithm, provided any three of them are given. Thus we have,

(i) **Amount** : $\log A = \log P + n \log (1 + i)$

(ii) **Principal** : $\log P = \log A - n \log (1 + i)$

(iii) Rate of interest : $\log (1 + i) = \dfrac{\log A - \log P}{n}$

(iv) Period (no. of years) : $n = \dfrac{\log A - \log P}{\log (1 + i)}$

If the interest is compounded half yearly,

$$A = P\left(1 + \frac{i}{2}\right)^{2n}$$

i.e. $\log A = \log P + 2n \log (1 + i/2)$

If the interest is compounded quarterly,

$$A = \left(P\ 1 + \frac{i}{4}\right)^{4n}$$

i.e. $\log A = \log P + 4n \log (1 + i/4)$

Thus the general formula is, $A = P (1 + i/q)^{nq}$ where, q is the number of equal periods partitioned in a year.

Illustration 1 : What is the principal value of Rs. 1000, due in 2 years at 5% p.a. compound interest, according as the interest is paid (a) yearly (b) half yearly ?.

Solution : (a) When interest is paid yearly :

Given A = 1000; i = 5/100 = 0.05; n = 2; P = ?

$\log P = \log A - n \log (1 + i)$

$\log P = \log 1000 - 2 \log (1.05)$

$= 3 - 2 \times 0.0212 = 3 - 0.0424$

$= 2.9576$

$\therefore P = \text{Antilog } (2.9576) = \text{Rs. } 906.90.$

Thus principal value when yearly compounded = Rs. 906.90

(b) When interest is paid half-yearly :

Given A = 1000; i/2 = 0.05/2 = 0.025; 2n = 2 x 2 = 4

$A = P (1 + i/2)^{2n}$

$\log P = \log A - 2n \log (1 + i/2)$

$\log P = \log 1000 - 4 \log (1.025)$

$= 3 - 4 \times 0.0107 = 3 - 0.0428$

$= 2.9572$

$\therefore P = \text{Antilog } (2.9572) = 906.10.$

Illustration 2 : Find the compound interest on Rs. 8000 for 4 years if interest is payable half yearly for the first three years at the rate of 8% p.a. and for the 4th year, the interest is payable quarterly at the rate of 6% p.a.

Solution : When the interest is paid half-yearly for 3 years.

Given P = Rs. 8000; n = 3; i = 8/100 = 0.08; i/2 = 0.04; 2n = 2 x 3 = 6.

$$A = P(1 + i/2)^{2n}$$

$$\log A = \log P + 2n \log (1 + i/2)$$

$$\log A = \log 8000 + 6 \log (1.04)$$

$$= 3.9031 + 6 \times 0.0170$$

$$= 3.9031 + 0.102 = 4.0051$$

$$\therefore A = \text{A.L.} (4.0051)$$

$$= 10120$$ (which is the principal for the 4th year).

Again when interest is paid quarterly in 4^{th} year.

P = 10120; i = 6/100 = 0.06; i/4 = 0.06/4 = 0.015; 4n = 4 x 1 = 4.

$$A = P(1 + i/4)^{4n}$$

$$\log A = \log P + 4n \log (1 + i/4)$$

$$\log A = \log 10{,}120 + 4 \log (1.015)$$

$$= 4.0051 + 0.0256 = 4.0307.$$

$$\therefore A = \text{A.L.} (4.0307) = 10740.$$

Compound interest at the end of 4^{th} year

= Rs. 10740 – Rs. 8000 = Rs. 2740.

Illustration 3 : Mr. Raman borrowed 10,000 from Mr. Vinoth, but could not repay any amount in the period of 5 years. After 5 years Mr. Vinoth sent a demand notice showing the sum of Rs.20000 due from him. At what rate of interest compounded annually did the latter lend his money ?.

Solution : Given A = Rs. 20,000; P = 10,000 ; n = 5; i = ?.

We know, $$\log (1 + i) = \frac{\log A - \log P}{n}$$

$$\log (1 + i) = \frac{\log 20{,}000 - \log 10{,}000}{5}$$

$$= \frac{4.3010 - 4.0000}{5}$$

$$= \frac{0.3010}{5} = 0.0602$$

$$(1 + i) = \text{Antilog} (0.0602) = 1.149$$

$$i = 1.149 - 1 = 0.149$$

Hence the required interest rate $= 0.149 \times 100 = 14.9\%$.

Illustration 4 : Find the rate of interest when Rs.500 amounts to Rs.800 in 10 years, compound interest being added quarterly.

Solution : Given P = Rs.500; A = Rs. 800; n = 10; i = ? 4n = 4 x 10 = 40.

$$A = P(1 + i/4)^{4n}$$

$$\log(1 + i/4) = \frac{\log A - \log P}{4n}$$

$$= \frac{\log 800 - \log 500}{4 \times 10}$$

$$= \frac{2.9031 - 2.6990}{40} = \frac{0.2041}{40}$$

$$\log(1 + i/4) = 0.005$$

$$\therefore (1 + i/4) = \text{Antilog}(0.005) = 1.012$$

$$i/4 = 1.012 - 1 = 0.012$$

$$\therefore i = 0.012 \times 4 = 0.048.$$

$$\therefore \text{Rate of interest} = 0.048 \times 100 = 4.8\%$$

Illustration 5 : In what time, will Rs. 1761 amount to Rs. 2142 at 4% p.a. compound interest ?.

Solution : Given P = Rs.1761; A = Rs.2142 ; i = 4/100 = 0.04

$$n = \frac{\log A - \log P}{\log(1 + i)}$$

$$= \frac{\log 2142 - \log 1761}{\log 1.04}$$

$$= \frac{3.3308 - 3.2457}{0.0170}$$

$$= \frac{0.0851}{0.0170} = 5.$$

$$\therefore \text{No. of years} = 5.$$

Illustration 6 : Find the number of years in which a sum of money will triple itself at a compound interest at 10% p.a.

Solution : Let P = Rs.100; A = Rs.300; i = 0.1 n = ?.

$$n = \frac{\log A - \log P}{\log(1 + i)}$$

$$= \frac{\log 300 - \log 100}{\log 1.1}$$

$$= \frac{2.4771 - 2.0000}{0.0414}$$

$$= \frac{0.4771}{0.0414} = 11.52$$

∴ The sum will triple itself in 11.52 years.

Illustration 7 : In a certain population, the annual birth and death rates per thousand are 39.4 and 19.4 respectively. Find the number of years in which the population will be doubled assuming that there is no immigration or emigration.

Solution :

Population growth rate per thousand = Birth rate – Death rate

= 39.4 – 19.4 = 20.

Let P = 100, then A = 200; i = 20/1000 = 0.02.

$$n = \frac{\log A - \log P}{\log (1 + i)}$$

$$= \frac{\log 200 - \log 100}{\log 1.02}$$

$$= \frac{2.3010 - 2.000}{0.0086}$$

$$= \frac{0.3010}{0.0086} = 35.$$

∴ The population will be doubled in 35 years.

Illustration 8 : Find the number of years in which a sum of money will double itself at 4% p.a. compound interest payable half-yearly ?.

Solution : Let P = Rs.100, then A = Rs. 200; i = 4/100 = 0.04; i/2 = 0.04/2 = 0.02

$$2n = \frac{\log A - \log P}{\log (1 + i/2)}$$

$$= \frac{\log 200 - \log 100}{\log 1.02}$$

$$= \frac{2.3010 - 2.000}{0.0086} = \frac{0.3010}{0.0086} = 35.$$

$$\therefore n = 35/2 = 17.5 \text{ years.}$$

Hence the sum will be doubled when interest is payable half yearly in 17.5 years.

Illustration 9 : A banker proposes to give Rs. 7000 at the end of 3 ¼ years from now. How much the banker ought to receive from the customers now in order to pay off the amount taking money to be worth 8% p.a. C.I.?.

Solution : Given A = Rs. 7000.; n = 3 ¼ years; i = 0.08; i/4 = 0.08/4 = 0.02.

$$A = P(1+i)^n$$

$$7000 = P(1+0.08)^3(1+0.02)^1$$

$$= P(1.08)^3(1.02)^1$$

$$7000 = P(1.285)$$

$$\therefore P = \frac{7000}{1.285} = \text{Rs. } 5447 \text{ (Approx.)}$$

Illustration 10 : A sum of money invested, compound interest payable yearly, amounts to Rs. 2704 at the end of the second year and to Rs.2812.16 at the end of the third year. Find the rate of interest and sum.

Solution : Given amount at the end of :

2^{nd} year i.e. $P(1+i)^2 = 2704$ .... (1)

3^{rd} year i.e. $P(1+i)^3 = 2812.16$ .... (2)

Dividing (2) by (1)

$$\frac{P(1+i)^3}{P(1+i)^2} = \frac{2812.16}{2704}$$

$$\text{i.e. } 1+i = 1.04$$

$$\therefore i = 1.04 - 1 = 0.04 = 4\%.$$

Applying the value of 'i' in equation (1)

$$\text{we get, } P(1+0.04)^2 = 2704$$

$$P(1.04)^2 = 2704$$

$$P(1.0816) = 2704$$

$$\therefore P = \frac{2704}{1.0816} = \text{Rs. } 2500$$

∴The principal is Rs. 2500 and the rate of interest is 4%.

Illustration 11 : The population of a city increases at the rate of 15 per thousand. What will be the population at the end of 5 years if the present population is 69,360 ?.

Solution : Given P = 69,360; n = 5; i = 15/1000 = 0.015.

$$A = P(1+i)^n$$

$\log A = \log P + n \log (1 + i)$

$= \log 69360 + 5 \log (1.015)$

$= 4.8411 + 5 \times 0.0064$

$= 4.8411 + 0.032 = 4.8731$

$\therefore A = \text{A.L.} (4.8731) = 74,660$

$\therefore$ Population at the end of 5 years = 74,660

Illustration 12 : The difference between the simple interest and compound interest on a sum is Rs.30 at 9% p.a. for 3 years. Find out the principal.

Solution : Let the principal be Rs. 100

S.I. on Rs. 100 for 3 years $= \frac{Pnr}{100} = \frac{100 \times 3 \times 9}{100} =$ Rs. 27

C.I. on Rs. 100 for 3 years $= P(1 + i)^n - P$

$= [100 (1.09)^3 - 100]$

= Rs. 129.5 – Rs. 100 = Rs. 29.5

Difference between C.I. and S.I. = Rs. 29.5 – Rs. 27 = Rs. 2.5

If the difference is Rs. 2.5, the principal is Rs. 100

If the difference is Rs.30, the principal is $= \frac{100 \times 30}{2.5} =$ Rs. 1200.

Illustration 13 : A person wishes to divide Rs. 32,525 between his son and daughter who are aged 15 and 18 respectively in such a way that their shares, if invested 5% C.I. should produce the same amount when they become 20 years of age. Find the share of each.

Solution : Let the share of son be Rs. **x**

The share of the daughter = (32, 525 – **x**)

The son has 5 years and the daughter has 2 years to become 20 years old.

Thus, **for son** : P = x; n = 5; i = 0.05; A = ?.

$A = P(1 + i)^n$

$= x(1 + 0.05)^5 = x(1.05)^5$ ....(1)

for daughter : P = (32525 – x); n = 2; i = 0.05; A = ?.

$A = (32,525 - x)(1 + 0.05)^2$

$= (32,525 - x)(1.05)^2$ ....(2)

Equating (1) and (2) we get

$x(1.05)^5 = (32,525 - x)(1.05)^2$

$$\frac{x \cdot (1.05)^5}{(1.05)^2} = (32{,}525 - x)$$

i.e. $x (1.05)^3 = (32, 525 - x)$

i.e. $1.1576\, x = 32525 - x$

$1.1576\, x + x = 32525;$

i.e. $2.1576x = 32525$

$$\therefore x = \frac{32525}{2.1576} = 15,074.6$$

∴ The share of the son = Rs. 15074.60

The share of the daughter = Rs. 32525 – Rs. 15,074.60

= Rs. 17,450.40.

University Examination Questions & Answers

Illustration 14 : Find the compound interest on Rs. 8000 for 5 years at 12% per annum

[B.Com., M.K.U. April '95

Solution : Given P = Rs. 8000; n = 5; i = 12/100 = 0.12

We know, $A = P (1 + i)^n$

$A = 8000 (1 + 0.12)^5$

$A = 8000 (1.12)^5$

Taking logarithm,

$\log A = \log P + n \log (1 + i)$

$\log A = \log 8000 + 5 \log (1.12)$

$= 3.9031 + 5 \times 0.0492$

$= 3.9031 + 0.2460 = 4.1491$

$A = \text{A.L.} (4.1491) = 14090.$

∴ Compound interest = A - P = Rs. 14090 – Rs. 8000

= Rs. 6090.

Alternative method : This problem can also be solved without using logarithm.

$A = P (1 + i)^n$

$= 8000 (1 + 0.12)^5$

$= 8000 (1.12)^5$

$= 8000 \times 1.762$

$A = 14096$

∴ **Compound interest** = A - P = Rs. 14096 – Rs. 8000

= Rs. 6096.

Illustration 17 : If Rs.5000 is deposited at the rate of 12% p.a. compound interest, in how many years the amount will be doubled ?.

[B.Com., M.K.U., April '95 modified]

Solution : Given P = Rs.5000; i = 0.12; n = ?.

Since the principal sum will be doubled A = Rs.5000 x 2 = Rs. 10,000.

$$n = \frac{\log A - \log P}{\log (1 + i)}$$

$$= \frac{\log 10{,}000 - \log 5000}{\log (1.12)}$$

$$= \frac{4.000 - 3.6990}{0.0492}$$

$$= \frac{0.301}{0.0492} = 6.12$$

Thus the sum of Rs. 5000 will be doubled itself in 6.12 years.

Illustration 18 : The difference between simple interest and compound interest is Rs. 384.60. Number of years = 4. Rate of interest = 10%. Find out the sum. **[B.Com., M.K.U. April 1999]**

Solution : Let the Principal be Rs. 100.

S.I. on Rs. 100 for 4 years = PNi = 100 x 4 x 0.1 = 40.

C.I. on Rs. 100 for 4 years = $P(1 + i)^n - P$

= $[100\ (1 + 0.1)^4 - 100]$ = 146.41 – 100 = 46.41

Difference between CI and SI = 46.41 – 40 = 6.41.

If the difference is Rs. 6.41, the Principal is Rs. 100.

If the difference is Rs. 384.60, the Principal = $\frac{100 \times 384.60}{6.41}$ = Rs. 6000

Illustration 19 : An Amount when invested for 3 years earns an extra sum of Rs. 98.56 under 4% p.a. compound interest than at the same rate of simple interest. Find the amount. **[B.Com., Modified M.K.U., April 2007]**

Solution : Let the principal be Rs. 100

S.I. on Rs. 100 for 3 years = PNi = 100 x 3 x 0.04 = Rs. 12.

C.I. on Rs. 100 for 3 years = $P(1 + i)^3 - 100$ = $100\ (1 + 0.04)^3 - 100$

= Rs. 12.49.

Difference between C.I. and S.I. = 12.49 – 12.00 = Rs. 0.49.

If the difference is Rs. 0.49, the Principal is Rs. 100

If the difference is Rs. 98.56, the principal $= \frac{100 \times 98.56}{0.49}$ = Rs. 20,114

Illustration 20 : Calculate compound interest on Rs. 7500 for 3 ½ years at 5.5 % p.a.

[B.Com., April 1999, April 2004, Nov. 2005]

Solution : Given : P = Rs. 7500; N = 3.5; R = 5.5 i.e. i = 0.055.

$$A = P(1+i)^3 \left(1+\frac{i}{2}\right)^1$$

$$= 7500\ (1+0.055)^3\ (1+0.0275)^1$$

$$= 7500\ (1.055)^3\ (1.0275)^1$$

$$= 7500 \times 1.1742 \times 1.0275 = 9049.$$

$$\therefore \text{C. I.} = A - P$$

$$= 9049 - 7500$$

$$= 1549.$$

Note : 3½ years can be taken as 7 half years. In that case

$$A = P\left(1+\frac{i}{2}\right)^7 = 7500\ (1+0.0275)^7 = \text{Rs. } 9069.$$

EXERCISE

I Choose the correct answer :

1) The compound interest on Rs. 30,000 for 3 years at 10% per annum is
(a) 9,930 (b) 9,000 (c) 39,930 (d) none

2) The compound interest on Rs. 5,000 for 2 years at 8% per annum where the interest is compounded half yearly is
(a) 832 (b) 408 (c) 849 (d) none

3) A sum of money invested in a Bank is Rs. 10,000. The Banker gives 12% per annum compound interest. The excess money received at the end of 3 years is
(a) 14,049 (b) 4,049 (c) 13,600 (d) 3,600

4) The compound interest on Rs. 1,000 for 2 years at 24% per annum where the interest is compounded once in 3 months is
(a) 1,262 (b) 1,594 (c) 124 (d) 594

5) The amount received at the end of 5 years if the sum invested is Rs. 20,000 at 9% compound interest is
(a) 30,772 (b) 10,772 (c) 9,000 (d) 29,000

10. PERCENTAGE

IMPORTANT FACTS AND FORMULAE

I. **Concept of Percentage** : By a certain *percent*, we mean that many hundredths. Thus, x percent means x hundredths, written as $x\%$.

To express x% as a fraction : We have, $x\% = \frac{x}{100}$.

Thus, $20\% = \frac{20}{100} = \frac{1}{5}$; $48\% = \frac{48}{100} = \frac{12}{25}$, etc..

To express $\frac{a}{b}$ as a percent : We have, $\frac{a}{b} = \left(\frac{a}{b} \times 100\right)\%$.

Thus, $\frac{1}{4} = \left(\frac{1}{4} \times 100\right)\% = 25\%$; $0.6 = \frac{6}{10} = \frac{3}{5} = \left(\frac{3}{5} \times 100\right)\% = 60\%$.

II. If the price of a commodity increases by R%, then the reduction in consumption so as not to increase the expenditure is

$$\left[\frac{R}{(100+R)} \times 100\right]\%$$

If the price of a commodity decreases by R%, then the increase in consumption so as not to decrease the expenditure is

$$\left[\frac{R}{(100-R)} \times 100\right]\%$$

III. **Results on Population** : Let the population of a town be P now and suppose it increases at the rate of R% per annum, then :

1. Population after n years $= P\left(1 + \frac{R}{100}\right)^n$.

2. Population n years ago $= \frac{P}{\left(1 + \frac{R}{100}\right)^n}$.

IV. **Results on Depreciation** : Let the present value of a machine be P. Suppose it depreciates at the rate of R% per annum. Then :

1. Value of the machine after n years $= P\left(1 - \frac{R}{100}\right)^n$.

2. Value of the machine n years ago $= \frac{P}{\left(1 - \frac{R}{100}\right)^n}$.

V. If A is R% more than B, then B is less than A by

$$\left[\frac{R}{(100+R)} \times 100\right]\%.$$

If A is R% less than B, then B is more than A by

$$\left[\frac{R}{(100-R)} \times 100\right]\%.$$

SOLVED EXAMPLES

Ex. 1. *Express each of the following as a fraction :*

(i) 56% *(ii) 4%* *(iii) 0.6%* *(iv) 0.08%*

Sol. (i) $56\% = \frac{56}{100} = \frac{14}{25}$. (ii) $4\% = \frac{4}{100} = \frac{1}{25}$.

(iii) $0.6\% = \frac{0.6}{100} = \frac{6}{1000} = \frac{3}{500}$. (iv) $0.08\% = \frac{0.08}{100} = \frac{8}{10000} = \frac{1}{1250}$.

Ex. 2. *Express each of the following as a decimal :*

(i) 6% *(ii) 28%* *(iii) 0.2%* *(iv) 0.04%*

Sol. (i) $6\% = \frac{6}{100} = 0.06$. (ii) $28\% = \frac{28}{100} = 0.28$.

(iii) $0.2\% = \frac{0.2}{100} = 0.002$. (iv) $0.04\% = \frac{0.04}{100} = 0.0004$.

Ex. 3. *Express each of the following as rate percent :*

(i) $\frac{23}{36}$ *(ii)* $6\frac{3}{4}$ *(iii) 0.004*

Sol. (i) $\frac{23}{36} = \left(\frac{23}{36} \times 100\right)\% = \left(\frac{575}{9}\right)\% = 63\frac{8}{9}\%$.

(ii) $0.004 = \frac{4}{1000} = \left(\frac{4}{1000} \times 100\right)\% = 0.4\%$.

(iii) $6\frac{3}{4} = \frac{27}{4} = \left(\frac{27}{4} \times 100\right)\% = 675\%$.

Ex. 4. *Evaluate :* (Bank P.O. 2003)

(i) 28% of 450 + 45% of 280

(ii) $16\frac{2}{3}\%$ *of 600 gm* $- 33\frac{1}{3}\%$ *of 180 gm* (R.R.B. 1998)

Sol. (i) 28% of 450 + 45% of 280 $= \left(\frac{28}{100} \times 450 + \frac{45}{100} \times 280\right) = (126 + 126) = 252$.

(ii) $16\frac{2}{3}\%$ of 600 gm $- 33\frac{1}{3}\%$ of 180 gm

$= \left[\left(\frac{50}{3} \times \frac{1}{100} \times 600\right) - \left(\frac{100}{3} \times \frac{1}{100} \times 180\right)\right]$ gm $= (100 - 60)$ gm $= 40$ gm.

Ex. 5. *(i) 2 is what percent of 50 ?* (S.S.C. 2000)

(ii) $\frac{1}{2}$ *is what percent of* $\frac{1}{3}$ *?* (S.S.C. 2002)

(iii) What percent of 7 is 84 ?

(iv) What percent of 2 metric tonnes is 40 quintals ?

(v) What percent of 6.5 litres is 130 ml ?

Sol. (i) Required percentage $= \left(\frac{2}{50} \times 100\right)\% = 4\%$.

(ii) Required percentage $= \left(\frac{1}{2} \times \frac{3}{1} \times 100\right)\% = 150\%$.

(iii) Required percentage $= \left(\frac{84}{7} \times 100\right)\% = 1200\%$.

(iv) 1 metric tonne = 10 quintals.

$\therefore$ Required percentage $= \left(\frac{40}{2 \times 10} \times 100\right)\% = 200\%.$

(v) Required percentage $= \left(\frac{130}{6.5 \times 1000} \times 100\right)\% = 2\%.$

Ex. 6. ***Find the missing figures :***

(i) ?% of 25 = 2.125 ***(ii) 9% of ? = 63*** ***(iii) 0.25% of ? = 0.04***

Sol. (i) Let $x\%$ of 25 = 2.125. Then, $\frac{x}{100} \times 25 = 2.125 \Leftrightarrow x = (2.125 \times 4) = 8.5.$

(ii) Let 9% of x = 6.3. Then, $\frac{9}{100} x = 6.3 \Leftrightarrow x = \left(\frac{6.3 \times 100}{9}\right) = 70.$

(iii) Let 0.25% of x = 0.04. Then, $\frac{0.25}{100} x = 0.04 \Leftrightarrow x = \left(\frac{0.04 \times 100}{0.25}\right) = 16.$

Ex. 7. ***Which is greatest in $16\frac{2}{3}\%$, $\frac{2}{15}$ and 0.17 ?***

Sol. $16\frac{2}{3}\% = \left(\frac{50}{3} \times \frac{1}{100}\right) = \frac{1}{6} = 0.166$, $\frac{2}{15} = 0.133$. Clearly, 0.17 is the greatest.

Ex. 8. ***If the sales tax be reduced from $3\frac{1}{2}\%$ to $3\frac{1}{3}\%$, then what difference does it make to a person who purchases an article with marked price of Rs. 8400 ?*** (S.S.C. 200...)

Sol. Required difference $= \left(3\frac{1}{2}\% \text{ of Rs. } 8400\right) - \left(3\frac{1}{3}\% \text{ of Rs. } 8400\right)$

$= \left(\frac{7}{2} - \frac{10}{3}\right)\% \text{ of Rs. } 8400 = \frac{1}{6}\% \text{ of Rs. } 8400$

$= \text{Rs. } \left(\frac{1}{6} \times \frac{1}{100} \times 8400\right) = \text{Rs. } 14.$

Ex. 9. ***An inspector rejects 0.08% of the meters as defective. How many will he examine to reject 2 ?*** (M.A.T. 2000)

Sol. Let the number of meters to be examined be x.

Then, 0.08% of $x = 2 \Leftrightarrow \left(\frac{8}{100} \times \frac{1}{100} \times x\right) = 2 \Leftrightarrow x = \left(\frac{2 \times 100 \times 100}{8}\right) = 2500.$

Ex. 10. ***Sixty-five percent of a number is 21 less than four-fifth of that number. What is the number ?***

Sol. Let the number be x.

Then, $\frac{4}{5} x - (65\% \text{ of } x) = 21 \Leftrightarrow \frac{4}{5} x - \frac{65}{100} x = 21 \Leftrightarrow 15x = 2100 \Leftrightarrow x = 140.$

Ex. 11. ***Difference of two numbers is 1660. If 7.5% of one number is 12.5% of the other number, find the two numbers.***

Sol. Let the numbers be x and y. Then, 7.5% of x = 12.5% of $y \Leftrightarrow x = \frac{125}{75} y = \frac{5}{3} y.$

Now, $x - y = 1660 \Rightarrow \frac{5}{3} y - y = 1660 \Rightarrow \frac{2}{3} y = 1660 \Rightarrow y = \left(\frac{1660 \times 3}{2}\right) = 2490.$

$\therefore$ One number = 2490, Second number $= \frac{5}{3} y = 4150.$

Ex. 12. ***In expressing a length 81.472 km as nearly as possible with three significant digits, find the percentage error.*** **(S.S.C. 1997)**

Sol. Error = (81.5 − 81.472) km = 0.028.

$\therefore$ Required percentage = $\left(\frac{0.028}{81.472} \times 100\right)\%$ = 0.034%.

Ex. 13. ***In an election between two candidates, 75% of the voters cast their votes, out of which 2% of the votes were declared invalid. A candidate got 9261 votes which were 75% of the total valid votes. Find the total number of votes enrolled in that election.*** **(S.S.C. 2003)**

Sol. Let the total number of votes enrolled be x. Then,

Number of votes cast = 75% of x. Valid votes = 98% of (75% of x).

$\therefore$ 75% of [98% of (75% of x)] = 9261

$\Leftrightarrow \left(\frac{75}{100} \times \frac{98}{100} \times \frac{75}{100} \times x\right) = 9261 \Leftrightarrow x = \left(\frac{9261 \times 100 \times 100 \times 100}{75 \times 98 \times 75}\right) = 16800.$

Ex. 14. ***Shobha's Mathematics Test had 75 problems i.e., 10 arithmetic, 30 algebra and 35 geometry problems. Although she answered 70% of the arithmetic, 40% of the algebra and 60% of the geometry problems correctly, she did not pass the test because she got less than 60% of the problems right. How many more questions she would have needed to answer correctly to earn a 60% passing grade?*** **(C.D.S. 2002)**

Sol. Number of questions attempted correctly = (70% of 10 + 40% of 30 + 60% of 35)
= (7 + 12 + 21) = 40.

Questions to be answered correctly for 60% grade = 60% of 75 = 45.

$\therefore$ Required number of questions = (45 − 40) = 5.

Ex. 15. ***If 50% of (x − y) = 30% of (x + y), then what percent of x is y?*** **(S.S.C. 2003)**

Sol. 50% of (x − y) = 30% of (x + y) $\Leftrightarrow \frac{50}{100}(x-y) = \frac{30}{100}(x+y)$

$\Leftrightarrow$ 5 (x − y) = 3 (x + y) $\Leftrightarrow$ 2x = 8y $\Leftrightarrow$ x = 4y.

$\therefore$ Required percentage = $\left(\frac{y}{x} \times 100\right)\% = \left(\frac{y}{4y} \times 100\right)\%$ = 25%.

Ex. 16. ***Mr. Jones gave 40% of the money he had, to his wife. He also gave 20% of the remaining amount to each of his three sons. Half of the amount now left was spent on miscellaneous items and the remaining amount of Rs. 12,000 was deposited in the bank. How much money did Mr. Jones have initially?***

Sol. Let the initial amount with Mr. Jones be Rs. x. Then,

Money given to wife = Rs. $\frac{40}{100}x$ = Rs. $\frac{2x}{5}$. Balance = Rs. $\left(x - \frac{2x}{5}\right)$ = Rs. $\frac{3x}{5}$.

Money given to 3 sons = Rs. $\left[3 \times \left(\frac{20}{100} \times \frac{3x}{5}\right)\right]$ = Rs. $\frac{9x}{25}$.

Balance = Rs. $\left(\frac{3x}{5} - \frac{9x}{25}\right)$ = Rs. $\frac{6x}{25}$.

Amount deposited in bank = Rs. $\left(\frac{1}{2} \times \frac{6x}{25}\right)$ = Rs. $\frac{3x}{25}$.

$\therefore \frac{3x}{25} = 12000 \Leftrightarrow x = \left(\frac{12000 \times 25}{3}\right) = 100000.$

So, Mr. Jones initially had Rs. 1,00,000 with him.

Short-cut Method : Let the initial amount with Mr. Jones be Rs. x.

Then, $\frac{1}{2}[100-(3\times 20)]\%$ of $(100-40)\%$ of $x = 12000$.

$$\Leftrightarrow \quad \frac{1}{2}\times\frac{40}{100}\times\frac{60}{100}\times x = 12000 \quad \Leftrightarrow \quad \frac{3}{25}x = 12000 \quad \Leftrightarrow \quad x = \left(\frac{12000\times 25}{3}\right) = 1000$$

Ex. 17. ***10% of the inhabitants of a village having died of cholera, a panic s during which 25% of the remaining inhabitants left the village. The population is reduced to 4050. Find the number of original inhabitants.*** **(S.S.C.**

Sol. Let the total number of original inhabitants be x.

Then, $(100-25)\%$ of $(100-10)\%$ of $x = 4050$

$$\Leftrightarrow \quad \left(\frac{75}{100}\times\frac{90}{100}\times x\right) = 4050 \quad \Leftrightarrow \quad \frac{27}{40}x = 4050 \quad \Leftrightarrow \quad x = \left(\frac{4050\times 40}{27}\right) = 6000.$$

$\therefore$ Number of original inhabitants = 6000.

Ex. 18. ***A salesman's commission is 5% on all sales upto Rs. 10,000 and 4% o sales exceeding this. He remits Rs. 31,100 to his parent company after deducting commission. Find the total sales.*** **(R.R.B. 2**

Sol. Let his total sales be Rs. x. Now, (Total Sales) – (Commission) = Rs. 31,100

$\therefore \quad x - [5\%$ of $10000 + 4\%$ of $(x-10000)] = 31100$

$$\Leftrightarrow \quad x - \left[\frac{5}{100}\times 10000 + \frac{4}{100}(x-10000)\right] = 31100 \quad \Leftrightarrow \quad x - 500 - \frac{(x-10000)}{25} = 311$$

$$\Leftrightarrow \quad x - \frac{x}{25} = 31200 \quad \Leftrightarrow \quad \frac{24x}{25} = 31200 \quad \Leftrightarrow \quad x = \left(\frac{31200\times 25}{24}\right) = 32500.$$

$\therefore$ Total sales = Rs. 32,500.

Ex. 19. ***Raman's salary was decreased by 50% and subsequently increased by 5 How much percent does he lose ?*** **(Hotel Management, 20**

Sol. Let original salary = Rs. 100.

New final salary = 150% of (50% of Rs. 100) = Rs. $\left(\frac{150}{100}\times\frac{50}{100}\times 100\right)$ = Rs. 7

$\therefore$ Decrease = 25%.

Ex. 20. ***Paulson spends 75% of his income. His income is increased by 20% and increased his expenditure by 10%. Find the percentage increase in his savings.***

Sol. Let original income = Rs. 100. Then, expenditure = Rs. 75 and savings = Rs.

$\therefore$ New income = Rs. 120, New expenditure = Rs. $\left(\frac{110}{100}\times 75\right)$ = Rs. $\frac{165}{2}$.

New savings = Rs. $\left(120-\frac{165}{2}\right)$ = Rs. $\frac{75}{2}$.

Increase in savings = Rs. $\left(\frac{75}{2}-25\right)$ = Rs. $\frac{25}{2}$.

$\therefore$ Increase% = $\left(\frac{25}{2}\times\frac{1}{25}\times 100\right)\% = 50\%$.

Ex. 21. ***The salary of a person was reduced by 10%. By what percent should h reduced salary be raised so as to bring it at par with his original salary?*** **(S.S.C. 200**

Sol. Let the original salary be Rs. 100. New salary = Rs. 90.

Increase on 90 = 10. Increase on 100 = $\left(\frac{10}{90}\times 100\right)\% = 11\frac{1}{9}\%$.

Ex. 22. ***When the price of a product was decreased by 10%, the number sold increased by 30%. What was the effect on the total revenue ?*** **(R.B.I. 2003)**

Sol. Let the price of the product be Rs. 100 and let original sale be 100 pieces.

Then, Total Revenue = Rs. (100×100) = Rs. 10000.

New revenue = Rs. (90×130) = Rs. 11700.

$\therefore$ Increase in revenue $= \left(\frac{1700}{10000} \times 100\right)\% = 17\%$.

Ex. 23. ***If the numerator of a fraction be increased by 15% and its denominator be diminished by 8%, the value of the fraction is $\frac{15}{16}$. Find the original fraction.***

Sol. Let the original fraction be $\frac{x}{y}$.

Then, $\frac{115\% \text{ of } x}{92\% \text{ of } y} = \frac{15}{16} \Rightarrow \frac{115x}{92y} = \frac{15}{16} \Rightarrow \frac{x}{y} = \left(\frac{15}{16} \times \frac{92}{115}\right) = \frac{3}{4}$.

Ex. 24. ***In the new budget, the price of kerosene oil rose by 25%. By how much percent must a person reduce his consumption so that his expenditure on it does not increase?***

Sol. Reduction in consumption $= \left[\frac{R}{(100 + R)} \times 100\right]\% = \left(\frac{25}{125} \times 100\right)\% = 20\%$.

Ex. 25. ***The population of a town is 1,76,400. If it increases at the rate of 5% per annum, what will be its population 2 years hence ? What was it 2 years ago ?***

Sol. Population after 2 years $= 176400 \times \left(1 + \frac{5}{100}\right)^2 = \left(176400 \times \frac{21}{20} \times \frac{21}{40}\right) = 194481$.

Population 2 years ago $= \frac{176400}{\left(1 + \frac{5}{100}\right)^2} = \left(176400 \times \frac{20}{21} \times \frac{20}{21}\right) = 160000$.

Ex. 26. ***The value of a machine depreciates at the rate of 10% per annum. If its present value is Rs. 1,62,000, what will be its worth after 2 years ? What was the value of the machine 2 years ago ?***

Sol. Value of the machine after 2 years

$=$ Rs. $\left[162000 \times \left(1 - \frac{10}{100}\right)^2\right]$ = Rs. $\left(162000 \times \frac{9}{10} \times \frac{9}{10}\right)$ = Rs. 131220.

Value of the machine 2 years ago

$=$ Rs. $\left[\frac{162000}{\left(1 - \frac{10}{100}\right)^2}\right]$ = Rs. $\left(162000 \times \frac{10}{9} \times \frac{10}{9}\right)$ = Rs. 200000.

Ex. 27. ***During one year, the population of a town increased by 5% and during the next year, the population decreased by 5%. If the total population is 9975 at the end of the second year, then what was the population size in the beginning of the first year?*** **(Hotel Management, 2003)**

Sol. Population in the beginning of the first year

$$= \frac{9975}{\left(1 + \frac{5}{100}\right)\left(1 - \frac{5}{100}\right)} = \left(9975 \times \frac{20}{21} \times \frac{20}{19}\right) = 10000.$$

Ex. 28. ***If A earns $33\frac{1}{3}$% more than B, how much percent does B earn less tha***

Sol. Required percentage = $\left[\frac{\left(\frac{100}{3}\right)}{\left(100+\frac{100}{3}\right)} \times 100\right]\% = \left(\frac{100}{400} \times 100\right)\% = 25\%$.

Ex. 29. ***If A's salary is 20% less than B's salary, by how much percent is B's sa more than A's ?***

Sol. Required percentage = $\left[\frac{20}{(100-20)} \times 100\right]\% = 25\%$.

Ex. 30. ***How many kg of pure salt must be added to 30 kg of 2% solution of salt water to increase it to a 10% solution ?*** (M.A.T. 2

Sol. Amount of salt in 30 kg solution = $\left(\frac{2}{100} \times 30\right)$ kg = 0.6 kg.

Let x kg of pure salt be added.

Then, $\frac{0.6+x}{30+x} = \frac{10}{100} \Leftrightarrow 60 + 100x = 300 + 10x \Leftrightarrow 90x = 240 \Leftrightarrow x = \frac{8}{3} = 2\frac{2}{3}$.

Ex. 31. ***Due to a reduction of $6\frac{1}{4}$% in the price of sugar, a man is able to buy 1 more for Rs. 120. Find the original and reduced rate of sugar.***

Sol. Let original rate be Rs. x per kg.

Reduced rate = Rs. $\left[\left(100 - \frac{25}{4}\right) \times \frac{1}{100} x\right]$ = Rs. $\frac{15x}{16}$ per kg.

$\therefore \quad \frac{120}{\frac{15x}{16}} - \frac{120}{x} = 1 \Leftrightarrow \frac{128}{x} - \frac{120}{x} = 1 \Leftrightarrow x = 8$.

So, original rate = Rs. 8 per kg.

Reduced rate = Rs. $\left(\frac{15}{16} \times 8\right)$ per kg = Rs. 7.50 per kg.

Ex. 32. ***In an examination, 35% of total students failed in Hindi, 45% failed English and 20% in both. Find the percentage of those who passed in both the subje***

Sol. Let A and B be the sets of students who failed in Hindi and English respecti

Then, $n(A) = 35$, $n(B) = 45$, $n(A \cap B) = 20$.

So, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = (35 + 45 - 20) = 60$.

$\therefore$ Percentage failed in Hindi or English or both = 60%.

Hence, percentage passed = (100 − 60)% = 40%.

Ex. 33. ***In an examination, 80% of the students passed in English, 85% in Mathema and 75% in both English and Mathematics. If 40 students failed in both the subje find the total number of students.***

Sol. Let the total number of students be x.

Let A and B represent the sets of students who passed in English and Mathema respectively.

Then, number of students passed in one or both the subjects

$= n(A \cup B) = n(A) + n(B) - n(A \cap B) = 80\%$ of x + 85% of x − 75% of x

$= \left(\frac{80}{100}x + \frac{85}{100}x - \frac{75}{100}x\right) = \frac{90}{100}x = \frac{9}{10}x.$

$\therefore$ Students who failed in both the subjects = $\left(x - \frac{9x}{10}\right) = \frac{x}{10}$.

So, $\frac{x}{10} = 40$ or $x = 400$. Hence, total number of students = 400.

EXERCISE 10

(OBJECTIVE TYPE QUESTIONS)

Directions : *Mark (✓) against the correct answer* :

1. The ratio 5 : 4 expressed as a percent equals : **(S.S.C. 2000)**
(a) 12.5% (b) 40% (c) 80% (d) 125%

2. 3.5 can be expressed in terms of percentage as : **(R.R.B. 1998)**
(a) 0.35% (b) 3.5% (c) 35% (d) 350%

3. Half of 1 percent written as a decimal is : **(S.S.C. 1999)**
(a) 0.005 (b) 0.05 (c) 0.02 (d) 0.2

4. What is 15 percent of Rs. 34 ? **(I.M.T. 2002)**
(a) Rs. 3.40 (b) Rs. 3.75 (c) Rs. 4.50 (d) Rs. 5.10

5. 63% of $3\frac{4}{7}$ is :
(a) 2.25 (b) 2.40 (c) 2.50 (d) 2.75

6. 88% of 370 + 24% of 210 − ? = 118 **(Bank P.O. 2003)**
(a) 256 (b) 258 (c) 268 (d) 358

7. 860% of 50 + 50% of 860 = ? **(R.B.I. 2003)**
(a) 430 (b) 516 (c) 860 (d) 960

8. 45% of 750 − 25% of 480 = ? **(Bank P.O. 2002)**
(a) 216 (b) 217.50 (c) 236.50 (d) 245

9. 40% of 1640 + ? = 35% of 980 + 150% of 850 **(S.B.I.P.O. 1997)**
(a) 372 (b) 842 (c) 962 (d) 1052

10. 218% of 1674 = ? × 1800
(a) 0.5 (b) 4 (c) 6 (d) None of these

11. 60% of 264 is the same as : **(Hotel Management, 2001)**
(a) 10% of 44 (b) 15% of 1056 (c) 30% of 132 (d) None of these

12. 270 candidates appeared for an examination, of which 252 passed. The pass percentage is :
(a) 80% (b) $83\frac{1}{2}\%$ (c) $90\frac{1}{3}\%$ (d) $93\frac{1}{3}\%$

13. 5 out of 2250 parts of earth is sulphur. What is the percentage of sulphur in earth ?
(a) $\frac{11}{50}$ (b) $\frac{2}{9}$ (c) $\frac{1}{45}$ (d) $\frac{2}{45}$

14. What percent ...

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ANSWERS

I Choose the correct Answer : (1) b (2) a (3) c (4) c (5) d

II Answer the following : (1) 31875 (2) 1,19,320 (3) 19,975 (4) 3432 (5) 57,333 (6) 3137 (7) 500 (8) 3112.46 (9) .2755 (10) 3498 (11) 310.30 (12) 800 (13) 1000 (14) 46,925 (15) 1335.58 (16) 23,519 (17) 15,644 (18) 48,620 (19) 7064 (20) 174.17 (21) 266 (22) 7000 (23) 8% (24) 46,000 (25) 6989 (26) 5918.

4.6. DISCOUNT

Discount is an allowance or a reduction from a stated amount. Discount allowed by the wholesaler to the retailer, discount allowed to debtors to encourage prompt payment, issue of securities at a price below the nominal value to attract the prospective investors, amount charged by a bank at the time of discounting a bill of exchange etc. are the examples of discount.

4.6.1. Trade Discount : Trade discount is normally allowed by a wholesaler to a retailer at a fixed percentage of list price or catalogue price to promote sales. It is generally based on the quanitity of goods purchased. It is recorded in the invoice but the sale and purchase transactions are recorded at net invoice price i.e. list price less trade discount. Hence no entry is made for trade discount in accounting records. For example, if A sells goods to B for Rs.1000 at a trade discount of 10%, the purchase price for B is only Rs.900 and sale price for A is Rs.900. It means that the trade discount allowed or received is not at all recorded in the books and it is only the net amount which is recorded on purchases or sales.

4.6.2. Cash Discount : Cash discount is offered to customers to encourage prompt payment. It is generally allowed by the sellers to the debtors. Cash discount is not shown in the invoice. **It is calculated at certain percentage on amount due based on the period of time within which payment is made. It is duly recorded in accounting books by opening discount account. Cash discount allowed is a loss and cash discount received is a gain.**

Illustration 1 : The catalogue price of goods sold by a wholesaler to a retailer is Rs.20,000. The wholesaler allows a trade discount of 15% and also provides 5% cash discount if the account is settled in 3 months. Find what is the net price of the goods sold ?.

Solution :

		Catalogue price of goods sold	=	Rs. 20,000
Less trade discount at 15%	=	$20{,}000 \times \frac{15}{100}$	=	Rs. 3,000
		Invoice price	=	Rs. 17,000
Less cash discount at 5%	=	$17{,}000 \times \frac{5}{100}$	=	Rs. 850
		∴ Net cash price	=	Rs. 16,150

Illustration 2 : A producer sells a sofa set listed at Rs. 12,000 to a retailer allowing a trade discount of 20% and cash discount 10%. The retailer in turn sells it allowing 10% discount and gaining 25% profit on the cost. Find the sales price and also find out the marked price of the retailer.

Solution :

List price of the producer		= Rs. 12,000
Less trade discount at 20%	$= 12{,}000 \times \frac{20}{100}$	= Rs. 2,400
∴ Invoice price		= Rs. 9,600
Less cash discount at 10%	$= 9{,}600 \times \frac{10}{100}$	= Rs. 960
Cash price to the producer or cost price to the retailer		= Rs. 8,640
Add profit earned by retailer at 25%	$= 8{,}640 \times \frac{25}{100}$	= Rs. 2,160
Selling price of the retailer		= Rs. 10,800
Add discount at 10% on marked price	$= 10{,}800 \times \frac{10}{90}$	= Rs. 1,200
∴ Marked price of the retailer		= Rs. 12,000

Illustration 3 : A merchant keeps the catalogue price of a pumpset 60% above cost, but gives 20% discount. If he makes a profit of Rs. 1400 per pumpset, find its cost price and catalogue price.

Solution :

Let the cost price of the pumpset be		= Rs. 100
Add profit on cost price at 20%	$= 100 \times \frac{60}{100}$	= Rs. 60
∴Catalogue price		= Rs. 160
Less trade discount at 20%	$= 160 \times \frac{20}{100}$	= Rs. 32
∴Invoice price or sale price		= Rs. 128
Profit on sale of pumpset	= Rs. 128 – Rs.100	= Rs. 28
If the profit is Rs.28, the cost price		= Rs. 100
If the profit is Rs. 1400, the cost price	$= \frac{100}{28} \times 1400$	= Rs. 5,000

$\therefore$ Catalogue price $= $ Cost price + Profit on cost price

$$= 5000 + \left(5000 \times \frac{60}{100}\right)$$

$$= 5000 + 3000$$

$$= \text{Rs. } 8000$$

Ans :

Cost price = Rs. 5000

Catalogue price = Rs. 8000

4.6.3. Present worth : **The present worth of a sum due sometime later is the amount that can be paid now as a present payment of the later. If a person has to** receive Rs.110 a year hence, he would not lose anything by accepting Rs.100 now. In this case, Rs.110 is called the **Sum due**, Rs.100 is called its **Present worth** and Rs.10 is called the **Discount** on Rs.110. The discount which is the difference between the sum due and the present worth is, in fact, the **interest** on the present worth. The terms **present worth, sum due and discount represent the principal, amount and simple interest**, respectively.

We derive the important formulae needed in this section with the following notations.

Let the present worth : P

Interest rate : i where, $i = \frac{R}{100}$

Period in years : n

Face value of the bill is given by

$$A = P + Pni$$

$$\therefore A = P(1 + ni)$$

Present worth is given by $P = \frac{A}{1 + ni}$

Illustration 1 : Find the present worth of a bill of Rs. 13,200 due 6 months hence at 20% p.a. simple interest.

Solution : A = Rs. 13,200; n = 6/12 = 0.5; i = 20/100 = 0.2; ni = 0.5 x 0.2 = 0.1

$$P = \frac{A}{1 + ni}$$

$$= \frac{13{,}200}{1 + 0.1} = \frac{13{,}200}{1.1} = 12{,}000$$

Alternative method

Let the present worth be Rs. 100

Interest on Rs. 100 for 6 months at 20% p.a. = P n i

= 100 × 0.5 × 0.2 = Rs.10

∴ Sum due = Rs. 100 + 10 = Rs. 110.

If the sum due is Rs. 110, present worth = Rs.100.

If the sum due is Rs. 13,200, present worth = $13,200 \times \frac{100}{110}$ = 12,000

Present worth of the bill = Rs. 12,000.

Illustration 2 : The present worth of a certain sum of money due 6 months hence is Rs. 4500 at 12% p.a. What is the sum due?.

Solution : P = Rs. 4,500; n = 6/12 = 0.5; i = 12/100 = 0.12

$$n\,i = 0.5 \times 0.12 = 0.06$$

$$A = P\,(1 + n\,i)$$

$$A = 4500 \times (1 + 0.06)$$

$$= 4500 \times (1.06) = 4770$$

Sum due = Rs. 4770.

4.6.4. True Discount : The seller while selling goods on credit to the buyer, charges certain percentage of interest to the actual sale price for the period of credit allowed by him. Thus there is a difference between actual selling price and the amount due from the buyer to the extent of interest charged on the sale price. This difference is known as True Discount.

Assume that X sold goods on credit to Y for Rs. 15,000 at 12% interest p.a. for 6 months. Now, X will draw a bill on Y for Rs. 15,900 (Rs. 15,000 actual sale price + 900 interest for 6 moths). Here Rs. 15,900 is the face value of the bill, Rs. 15,000 is the present value or worth of the bill and the difference Rs. 900 is the true discount which is the simple interest on the present worth of the bill.

The formulae for calculating true discount is :

$$\mathbf{TD = P\,n\,i}$$

$$\text{Also, } TD = A - P$$

$$P = A - \frac{A}{1 + n\,i}$$

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$$P = \frac{A(1+ni) - A}{1+ni} = \frac{Ani}{1+ni}$$

Thus, $$TD = \frac{Ani}{1+ni}$$

where,
TD = True discount
A = Face value of the bill or amount due.
P = Present worth
n = Period in years
i = rate of interest

The true discount can also be computed as the difference between Banker's Discount and Banker's Gain.

$$TD = BD - BG$$

BD - Banker's discount
BG - Banker's gain.

The **true discount** can also be computed as,

$$TD = \frac{BG}{ni}$$

where, BG - Banker's Gain
n - discounting period in years
i - rate of interest.

4.6.5. Discounting a Bill of Exchange : Sometimes, the holder of the bills receivable does not keep the bill with him until the date of maturity. If he is in need of cash before the due date of the bill, he can get the bill discounted with the banker. The banker will charge interest at the prevailing rate on the face value of the bill from the date of discounting to the due date and pays cash after deducting such interest from the value of the bill. This is called **discounting the bill**. The banker would retain the bill for the remaining term and obtain its full amount from the drawee at maturity.

Due date : The date on which the amount of the bill becomes payable is called 'due date' or 'date of maturity'

Nominal due date : The date on which the actual term of a bill expires is called the Nominal Due date. For example, for a bill drawn on May 27 for 4 months, the nominal due date is Sep.27.

Legal due date : To calculate the due date, three days are added to the date of maturity of the period of the bill. Thus the additional three days allowed to the drawee ... 'Days of Grac... The date

obtained after adding these three extra days is called the Legal Due Date. This is the date on which actual payment must be made. Thus for a bill drawn on Feb.12 for 3 months, the nominal due date is May 12 and the legal due date is May 15.

4.6.6. Banker's Discount : Banker's discount is the interest charged by the bank on the face value of the bill for the period from date of discounting to the date of maturity. In other words, banker's discount is the amount charged by the bank at the time of discounting of the bill of exchange for discounting future cash flow to its present value. It is also called Mercantile Discount. Generally, Banker's discount is greater than true discount for the same period because true discount is the interest calculated on the present worth of the bill, whereas, banker's discount is the interest calculated on the face value of the bill or on the sum due. The banker's discount can be calculated as follows.

$$\textbf{Banker's Discount} = \left\{\begin{matrix}\text{Face value} \\ \text{of the bill}\end{matrix}\right\} \times \left\{\begin{matrix}\text{Period of} \\ \text{discount}\end{matrix}\right\} \times \left\{\begin{matrix}\text{Rate of} \\ \text{interest}\end{matrix}\right\}$$

Symbolically,

$$BD = Ani$$

where,

BD - Banker's discount

A - Face value of the bill

n - Period of discount in years

i - rate of interest

(or)

$$BD = TD + BG$$

where,

TD - True discount

BG - Banker's gain.

4.7. Banker's Gain : The difference between Banker's discount and true discount is known as Banker's Gain.

$$BG = BD - TD$$

where; BG - Banker's Gain

BD - Banker's discount.

TD - True discount.

From the above formula, we get the relation as

$$BG = BD - TD$$
$$= Ani - Pni$$
$$= (A - P)ni$$
$$BG = TD \times ni$$

Banker's gain can also be computed by

$$BG = BD - TD$$
$$= Ani - \frac{Ani}{1+ni}$$
$$= \frac{Ani(1 + ni - 1)}{1+ni} = \frac{A(ni)^2}{1+ni}$$

Thus, $$BG = \frac{A(ni)^2}{1+ni}$$

Also, $$BG = P(ni)^2 \qquad \left[\because P = \frac{A}{1+ni}\right]$$

Face value of the bill can also be computed as

$$A = \frac{BD}{ni} \qquad (\text{Since } BD = Ani)$$
$$= \frac{BD \times TD}{BG} \qquad \left(\text{Since } BG = TD \times ni \text{ and } ni = \frac{BG}{TD}\right)$$

Thus, Face value of bill (A) = $$\frac{BD \times TD}{BD - TD}$$

By definition, value of the bill after discount is given by

Value of the bill after discount $= A - Ani$

$= A(1 - ni)$

Hence, Face value of bill (A) = $$\frac{\text{Value of bill after discount}}{1 - ni}$$

Illustration 1 : X sold goods worth Rs.5000 to Y for 6 months credit and drew a bill on Y at 6% interest per annum. X, immediately discounted the bill with a banker at the same rate of interest and got the balance from the bank. Find (i) True discount, (ii) Banker's discount (iii) Amount received from the Bank and (iv) Banker's gain.

Solution :

Sales value of goods = Rs. 5,000

Add interest for 6 months at 6% p.a. $= 5,000 \times \frac{6}{100} \times \frac{6}{12}$ = Rs. 150

∴ Amount due or Face value of the bill = Rs. 5150

∴ (i) True discount = Rs. 5150 – Rs. 5000 = Rs. 150

$A = 5150; \quad n = 6/12 = 0.5; \quad i = 6/100 = 0.06$

(ii) Banker's discount = A n i

$= 5150 \times 0.5 \times 0.06$

= Rs. 154.50.

(iii) Amount received from bank = Face value of the bill – Banker's discount

= Rs. 5150 – Rs. 15.50 = Rs. 4995.50

(iv) Banker's gain = B D – T D

= Rs. 154.50 – Rs. 150

= Rs. 4.50.

Illustration 2 : Find the true discount on a bill of Rs. 5175 due 6 months hence, if the rate of interest is 7% p.a. Also find out (i) Banker's discount and (ii) Banker's gain.

Solution :

Given : $A = 5175; \; n = 6/12 = 0.5; \; i = 7/100 = 0.07$

Face value of the bill = Rs. 5175

Less present worth of the bill i.e. $= P = \frac{A}{1 + ni} = \frac{5175}{1 + 0.5 \times 0.07}$

$= \frac{5175}{1.035}$ = Rs. 5000

∴ True discount = Rs. 175

(ii) Banker's discount $= A\,n\,i$

$= 5175 \times 0.5 \times 0.07$

$=$ Rs. 181.13.

(iii) Banker's gain $= BD - TD$

$=$ Rs.181.13 − Rs. 175

$=$ Rs. 6.13.

Illustration 3 : A bill for Rs. 3745 was drawn on 10th February at 6 months' date and discounted on 5th May at the rate of 9% p.a. Find (i) Banker's discount (ii) True discount, (iii) Amount received from the Bank and (iv) Banker's gain.

Solution :

Legal due date of the bill = 10th August + 3 days = 13th August

Period of discount = 5th May to 13th Aug.

= 26 + 30 + 31 + 13 = 100 days.

A = Rs. 3745; n = 100/365; i = 9/100 = 0.09;

$$n\,i = \frac{100}{365} \times 0.09 = 0.02466.$$

(i) Banker's discount $= A\,n\,i$

$= 3745 \times 0.02466$

$=$ Rs. 92.34

(ii) True discount $= \dfrac{A\,n\,i}{1 + n\,i}$

$$= \frac{92.34}{1 + 0.0246} = \frac{92.34}{1.0246} = \text{Rs. } 90.12$$

(iii) Amount received from bank = Face value of the bill − Banker's discount

= Rs. 3745 − Rs. 92.34 = Rs. 3652.66.

(iv) Banker's gain $= BD - TD$

$=$ Rs. 92.34 − Rs. 90.12

$=$ Rs. 2.22.

Illustration 4 : A bill of Rs. 5200 drawn on June 7 for 3 months was discounted for Rs. 5094 at 12% p.a. On what date was it discounted ?.

Solution : Given A = Rs. 5200; n = 3/12 = 0.25; i = 12/100 = 0.12

Banker's discount = Rs. 5200 – Rs. 5094 = Rs. 106;

Let us first find out the discounting period

$$n = \frac{B\,D}{A\,i} \quad (\therefore B\,D = A\,n\,i)$$

$$= \frac{106}{5200 \times 0.12} = 0.17 \text{ yrs. or } 0.17 \times 365 = 62 \text{ days.}$$

∴ The bill was discounted 62 days before maturity.

Due date of the bill	=	June 7 + 3 months + 3 grace days = Sep. 10.
62 days before Sep.10	=	Sep. 10 - {(Sep.) 10 + (Aug.) 31 + (July) 21} = July 10
∴ Date of discounting	=	July 10.

Illustration 5 : A bill of Rs. 5,000 drawn on 6 months was discounted for Rs. 4,880 on July 8. If the rate of interest was 10% p.a. on what date was the bill drawn?

Solution : Given A = Rs. 5000; n = 6/12 = 0.5; i = 10/100 = 0.1

Banker's discount = Rs. 5000 – Rs. 4880 = Rs. 120

$$\text{Discounting period } (n) = \frac{B\,D}{A\,i}$$

$$= \frac{120}{5000 \times 0.1} = 0.24 \text{ yrs. or } 0.24 \times 365 = 88 \text{ days.}$$

Due date of the bill is 88 days after July 8:

∴ Due date	=	(July) 23 + (Aug.)31 + (Sep.) 30 + (Oct.) 4 = Oct. 4
∴ Nominal Due date	=	Oct. 4 – 3 grace days = Oct. 1.
∴ Date of drawing the bill	=	Oct. 1 – 6 months = April 1.

Illustration 6 : A bill of Rs. 12,775 was drawn on June 10 for 4 months. It was discounted at a bank on July 20 for Rs. 12,537. Find the rate of interest charged by the bank.

Solution :

Due date of the bill	=	June 10 + 4 months + 3 grace days = Oct. 13
Period of discounting	=	(July) 11 + (Aug.) 31 + (Sep.) 30 + (Oct.) 13
	=	85 days.

Banker's discount = Rs. 12,775 – Rs. 12, 537 = . Rs. 238.

A = 12,775; n = 85/365 = 0.233; B D = 238; I = ?.

$$I = \frac{B\,D}{A\,n} \qquad [\because BD = AnI]$$

$$= \frac{238 \times 365 \times 100}{12{,}775 \times 85} = 8\%\text{ p.a.}$$

∴ Rate of interest charged by the bank = 8% p.a.

Illustration 7 : The bånker's gain on a bill due after 6 months at 12% p.a. is Rs. 36. Find the (i) True discount, (ii) Banker's discount and (iii) Face value of the bill.

Solution :

Given : B G = Rs. 36; n = 6/12 = 0.5; i = 0.12.

(i) True discount $= \frac{B\,G}{n\,I} \qquad [\because BG = TD \times nI]$

$$= \frac{36}{0.5 \times 0.12} = \text{Rs. } 600$$

(ii) Banker's Discount = B G + T D

= Rs. 36 + Rs. 600 = Rs. 636.

(iii) Face value of the bill $= \frac{BD \times TD}{BD - TD}$

$$= \frac{636 \times 600}{636 - 600} = \frac{3{,}81{,}600}{36} = \text{Rs. } 10{,}600.$$

Illustration 8 : The banker's discount and true discount on a bill due 6 months hence are Rs. 275 and Rs. 250 respectively. Find the amount of the bill and rate of interest.

Solution : Given : B D = Rs. 275; T D = Rs. 250;

B G = Rs. 275 – Rs. 250 = Rs. 25; n = 6/12 = 0.5.

Amount of the bill $= \frac{BD \times TD}{BD - TD}$

$$= \frac{275 \times 250}{275 - 250} = \frac{68750}{25} = \text{Rs.}2750$$

$$i = \frac{BD}{An} \qquad [\because BD = Ani]$$

$$= \frac{275 \times 100}{2750 \times 0.5} = 20\% \text{ p.a.}$$

Illustration 9 : Find the present worth of a bill drawn on April 15 at 3 months ar discounted on May 6 at 7.5% p.a. for Rs. 39,991.

Solution :

Due date of the bill = Apr. 15 + 3 months + 3 grace days = July 18.

Date of discounting = May 6.

∴ Period of discounting (n) = (May) 25 + (June) 30 + (July) 18 = 73 days

Let the face value of the bill be Rs. 100.

n = 73/365 = 0.2; i = 7.5/100 = 0.075; A = 100.

$BD = Ani = 100 \times 0.2 \times 0.075$ = Rs. 1.50.

∴ Discounted value = Rs. 100 – Rs. 1.50 = Rs. 98.50

For discounted value of Rs. 98.50, face value = Rs. 100.

∴ For discounted value of Rs. 39,991, face value

$$= \frac{100}{98.50} \times 39{,}991 = \text{Rs. } 40{,}600$$

Let present worth the bill be Rs. 100.

∴ Sum due = Rs.100 + (100 x 0.2 x 0.075)

= Rs.100 + Rs. 1.50 = Rs. 101.50.

For the sum of due Rs. 101.50, present worth = Rs. 100

For sum due Rs. 40,600, present worth $= \frac{100}{101.50} \times 40{,}600$ = Rs. 40,000

Present worth of the bill = Rs. 40,000

Illustration 10 : The difference between true discount and banker's discount on a bill, due 6 months hence at 12% p.a. is Rs. 72. Find (i) the amount of the bill and (ii) Banker's discount.

Solution : Let the present worth of the bill be Rs. 100.

P W = 100; n = 6/12 = 0.5; i = 12/100 = 0.12

True discount = 100 x 0.5 x 0.12 = Rs. 6.

The sum due = PW + TD = Rs. 100 + Rs.6 = Rs. 106.

BD = A n i = 106 x 0.5 x 0.12 = Rs. 6.36.

The difference between Banker's discount and True discount

= Rs. 6.36 – Rs. 6.00 = Rs. 0.36.

If the difference (B D - T D) is Rs. 72, the sum is

$$= \frac{106}{0.36} \times 72 = \text{Rs. } 21{,}200$$

(i) The amount of the bill = Rs. 21,200.

(ii) Banker's discount = A n i

= Rs. 21, 200 x 0.5 x 0.12 = Rs. 1272.

Illustration 11 : The banker's discount on a certain bill is Rs.710 and the true discount is Rs. 660 at the same rate. What is the amount of the bill ?.

Solution : Given : B D = Rs. 710; T D = Rs. 660

$$\text{Amount of the bill} = \frac{BD \times TD}{BD - TD}$$

$$= \frac{710 \times 660}{710 - 660} = \frac{4{,}68{,}600}{50} = \text{Rs. } 9372$$

Illustration 12 : A bill was accepted on 8th Jan., 1985 for 4 months. It was discounted on 11th Feb. to meet a bill for Rs. 9900 due on that date. If the rate of discount was 4% p.a., what was the face value of the first bill ?.

Solution :

Due date of the bill = Jan. 8 + 4 months + 3 grace days = May 11

Discounting period = Feb. 11 to May 11. = 3 months.

$n = 3/12 = 0.25$; $i = 4/100 = 0.04$; $ni = 0.25 \times 0.04 = 0.01$.

$$\text{Face value of the bill (A)} = \frac{\text{Value of the bill after discounting}}{1 - ni}$$

$$= \frac{9900}{1 - 0.01} = \frac{9900}{0.99} = \text{Rs.}10{,}000$$

1

THEORY OF SETS

1.0. Introduction

Set theory is the foundation of modern mathematics, which was found by George Cantor (1845-1918). We often talk about sets of objects such as **set of students, set of sales divisions, set of Industries, set of Income etc.** In set we deal with a group of **objects** which can be defined in terms of their distinct characteristics, magnitudes etc. The terms aggregate, collection and population have the same meaning as that of set.

1.1. Definition : A set is a well-defined collection of objects. We may describe a set as a collection of objects viewed as a whole.

The **objects** forming a set are called **elements** or **members** of the set. A set may or may not always contain elements having a common property.

Examples of sets : (i) The set of all students in a class room. Here individual students are called elements or members and collection of all students is called a set.

(ii) The set of Public Sector Industries in India. This is a set of well defined objects, because public sector is a well defined concept.

Generally we denote sets by capital letters such as A, B, C .... and the elements by small letters such as a, b, c, ... If 'a' is an element of a set S, we denote it symbolically, **a ∈ S** (Read as 'a' is an element of 'S' or 'a' belongs to 'S'). If on the other hand, 'a' is not an element of 'S' we write a ∉ S (read as 'a' is not an element of 'S').

1.2. Description of sets : A set can be described in any one of the following three ways:

(i) listing elements
(ii) stating common property
(iii) precise statement

1.2.1. By listing the elements : If there is no obvious connection between the objects forming a set, it is necessary to list all the elements of the set.

A set is given by listing its elements in a row, separating them by commas and enclosing all of them in brackets { }

Thus, if a set S consists of five letters a, b, c, d and e, we write S = {a, b, c, d, e }. This method of representation is called **Roaster** or **Tabulation** or **Enumeration** method.

Examples : (i) The set of positive integers less than 10 is given by
S = {1, 2, 3, 4, 5, 6, 7, 8, 9}

(ii) The set of odd natural numbers is given by :
S = { 1, 3, 5, 7, 9, ... }

Illustration 1 : Express the following sets by the Roaster method.

(a) The set of all integers between 6 and 11.
(b) The set of all odd integers greater than 6 but less than 14.

Solution : (a) : S = {7, 8, 9, 10}
(b) : A = {7, 9, 11, 13} 1, 3, 5, (7, 9, 11, 13) 15

1.2.2. By stating the common property : In case the elements of a set have a common property, we can use this property to determine the elements of the set. This method is known as **Set builder or Rule defining** method.

Examples : (i) S = { x | x is an even positive integer less than 10 }
(Read symbol "|" as "such that"). As a whole it can be read as the set of all x such that x is an even positive integer less than 10.
Thus, S = {2, 4, 6, 8}

(ii) S = {x | x is a vowel in English alphabet}.
Thus, S = {a, e, i, o, u}

(iii) C = {x | x is the components of cost of production}
Thus, C = {material, labour, overheads}

1.2.3. By precise statement : A set may be represented by means of a precise statement in words.

Examples : (i) Natural numbers 1, 2, 3, 4, 5, 6 may be represented by this method as below.
S = set of first six natural numbers.

(ii) Elephant, Tiger, Lion can be represented by :
S = Set of three wild animals.

1.3. Type of sets :

1.3.1. Null set : A set having no element is called a Null set or an Empty set or a Void set. It must be noted that a **Null set is a subset of every set.** It is denoted by "{ }" or the Greek letter ϕ read as " phi " ϕ

Examples : (1) The set of all women presidents in India till 2006.
(2) Set of all persons whose height is greater than 20 feet.
(3) A = {x | x ∈ N, x is a perfect square and 26 < x < 35}.

Note : A Null set is a finite set.

1.3.2. Singleton set : A set having only one element is called a singleton set.

Examples : (i) The set of mothers of a person

(ii) A = {2}

(iii) The set of all roots of the equation $x - 3 = 0$.

(iv) A = { x | x is a straightline passing through two given points }

1.3.3. Subsets : The set A is called a subset of the set S if every element of A is also an element of S. Thus, if A is a subset of S, then we say that A is contained in S. It is denoted by $A \subseteq S$ which may be read a **"A is a subset of S"** or **"A is included in S"**.

Sometimes this relationship is written as $S \supseteq A$ and is read as **"S is a super set of A" or "S contains A"**.

It may be noted that $A \subseteq S$ means that every element of A is also an element of S i.e. $x \in A \Rightarrow x \in S$. (Read the symbol $\Rightarrow$ as "implies that").

Examples : (1) The set of students of commerce department is a subset of the set of all students in a college.

A = {x | x is a student in C.P.A. College}

B = {x | x is a commerce student in C.P.A. College}

Then $B \subseteq A$.

(2) The set of even positive numbers {2, 4, 6, ... } is a subset of the set of all positive integers {1, 2, 3 4, 5, 6 ..... }.

(3) If 'B' is a set representing the sale (in units) of a commodity and 'M' be the set representing the entire quantity (in units) manufactured, then B is the subset of M i.e. $B \subseteq M$.

(4) A = { 1, 2, 3, 4 }, B = { 5, 8. 1. 4, 6, 2, 7, 3 }

then $A \subseteq B$.

1.4. Properties of subsets :

(1) The empty set is a subset of every set.

(2) Every set is a subset of itself i.e. $A \subseteq A$.

eg. A = { 1, 2, 3 } Clearly $A \subseteq A$

(3) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

eg. A = { 1, 2, 3 }; B = { 1, 2, 3, 4, 5 }; C = { 1, 2, 3, 4, 5, 6 }

Illustration 2 : When A = { 2, 7, 9, 15 }; B = { 20, 17, 2, 9, 15, 1 } and C = {30, 19, 1, 15, 22, 17, 9, 20, 2, 7}, analyse whether $A \subseteq C$ is correct?

Solution : Given A = {2, 7, 9, 15}; B = {20, 17, 2, 7, 9, 15, 1}

$\therefore A \subseteq B$

Given B = {20, 17, 2, 7, 9, 15, 1}; C = {30, 19, 1, 15, 22, 17, 9, 20, 2, 7}

$\therefore B \subseteq C$

we know $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

$\therefore$ The statement $A \subseteq C$ is correct.

1.5. Remark : An element x of a set A cannot be a subset of A but the singleton set {x} is a subset of A.

i.e. $x \not\subset A$ but $\{x\} \subseteq A$

1.6. Number of subsets of a set : If there are 'n' elements in a set; there will be 2^n subsets for the set. For example, if there are 2 elements in a set, the total no. of subsets will be $2^2 = 4$. If there are 3 elements in a set, the total number of subsets will be $2^3 = 8$ and so on.

Illustration 3 : Write down the subsets of the set {5, 6, 7}

$2^3 = 8$

Solution : The subsets of the set A = {5, 6, 7} are :

{5, 6, 7}, {5, 6} {5, 7}, {6, 7}, {5}, {6}, {7} and ϕ.

1.6.1. Power set : The set of all distinct subsets of a set is called a power set. It is symbolically denoted by ρ (A) read as power set of A. In the above illustration, the power set of A can be written as follows.

ρ (A) = {{5, 6, 7}, {5, 6}, {5, 7}, {6, 7}, {5}, {6}, {7}, ϕ}

1.7. Proper subsets : A set S is called a proper subset of B if S is a subset of B and B has atleast one element which is not in S. It is symbolically written as $S \subset B$.

Examples : S = {1, 2, 3, 5}

B = {1, 2, 3, 5, 9}

Here $S \subset B$ because the element 9 is not in S.

{B = 5}

Illustration 4 : If A = {3, 5, 7} and B = {x | 2x = 10}. Which of the following is true?

a) $B \subset A$ b) A = B c) $A \subset B$

Solution : Given A = { 3, 5, 7 }

B = { x | 2x = 10 } = { 5 } {2x = 10, $\therefore$ x = 5}

We can conclude that (a) $B \subset A$ is only true.

Illustration 5 : Write proper subsets of a) {0, 1} and b) {a, b, c}

Solution : a) Proper subsets of the set {0, 1} are {0}, {1} and ϕ

b) Proper subsets of {a, b, c} are {a}, {b}, {c}, {a, b}, {a, c}, {b, c} and ϕ.

1.8. Equality of sets : Two sets A and B are said to be equal if $A \subset B$ and $B \subset A$ i.e. every element of A is an element of B and every element of B is an element of A. Thus, A = B implies that both have the same members.

Examples : (i) A = {1, 2, 3}; B = {3, 1, 2} $\therefore$ A = B.

(ii) A = {3, 4}, B = {4, 3} and

C = {x | x is an integer and $2 \le x \le 5$} = {3, 4}

$\therefore$ A = B = C

Illustration 6 : Verify whether A = B from the following :

(a) A = {1, 2, 3} and B = {x | (x - 1) (x - 2) (x - 3) = 0}

Solution : Given A = {1, 2, 3}

B = {x | (x - 1) (x -2) (x - 3) = 0}

Solving x - 1 = 0 $\therefore$ x = 1; x - 2 = 0 $\therefore$ x = 2; x - 3 = 0 $\therefore$ x = 3

$\therefore$ B = {1, 2, 3}. Thus we conclude that A = B

(b) A = {1, 3, 5, 7, 9} and B = {x | x is a positive odd one digit number}

Solution : Given

A = {1, 3, 5, 7, 9} and

B = {x | x is a positive odd one digit number}

= {1, 3, 5, 7, 9} $\therefore$ A = B.

1.9. Finite and Infinite sets :

1.9.1. Finite set : A set is said to be finite, if it has a finite number of elements. Here the elements can be counted by a definite number.

Examples : (i) The set of students in a class is a finite set.

(ii) A = {4, 7, 9, 15, 22, 30} is a finite set, for, A has 6 elements.

(iii) A = {x | x is an even positive integer $\le$ 100}, is a finite set, for A has 50 elements.

1.9.2. Infinite set : A set which is not finite is called an Infinite set. Here the number of elements of the set can not be definitely known.

Examples : i) A = {1, 2, 3, ... } i.e the set of all natural numbers.

ii) A = {The set all human wants}

Thus X = {2, 4, 6, 8, 10} **is a finite set, where as**

Y = {2, 4, 6, 8, 10, ...} **is an infinite set.**

Illustration 7 : State whether the following sets are finite or infinite or empty (i) X = {1, 2, 3, ..., 500}, (ii) A = {2^n | n is a positive integer} and (iii) Y = {y | y = a^2, a is an integer}.

Solution : (i) X = {1, 2, 3, ..., 500}. Since there are only 500 elements in the set X, it is a finite set.

(ii) A = {2^n | n is a positive integer}

= {$2^1, 2^2, 2^3, 2^4$, ...}

= {2, 4, 8, 16, ...} is an infinite set.

(iii) Y = {y | y = a^2, a is an integer}

= {$0^2, 1^2, 2^2, 3^2, 4^2, 5^2$, ...}

= {0, 1, 4, 9, 16, 25, ...} is an infinite set

Note : 0 is an integer. But it is neither positive nor negative integer.

1.10. Universal set :

The set, which contains all the sets under consideration as subsets, is called an universal set. It is usually denoted by U or X.

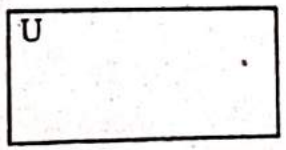
eg. (1) If we consider the set of commerce students, the set of Economics students and the set of Science students in a college, then the universal set is the set of all students in the college.

(2) A set of cost Accountants in India may be considered as a universal set for a set of Associate members of I.C.W.A.I.

1.11. Venn diagram :

English logician John Venn (1839-1923) introduced the idea of representing a set **with geometric intuition. Such pictorial representation is known as Venn diagram. Thus venn diagram is a pictorial representation of different types of sets.** Several set relations can be easily shown by these diagrams. The universal set, say U or X is usually represented by a **rectangle** and other sets say, A, B, C are shown through **circles or closed curves inside the universal set**

These circles **intersect each other,** if there are any common elements amongst them and if there are no common elements then they are shown separately as **disjoint circles.** The venn diagrams are useful to illustrate the set relations and set-operations.



Rectangle represents universal set.

1.12. Set operations :

Just like the fundamental mathematical operations i.e. **addition, subtraction, multiplication and division** there are some basic operations on sets. They are **union ∪, intersection ∩ and difference ~.** Let us explain set operations with Venn diagrams.

1.12.1. Union of sets : It is the set of all elements which belong to sets A or B or both A and B. The union of two sets A and B is denoted by A ∪ B and is read as "A union B". (read also as cup)

A ∪ B = { x | x ∈ A or x ∈ B or both }

Examples :

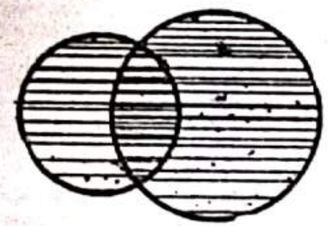
(1) A = {1, 2, 3, 4, 5}; B = {6, 7}
∴ A ∪ B = {1, 2, 3, 4, 5, 6, 7}

(2) A = {1, 2, 3, 4}; B = {2, 3, 4, 5, 6}
∴ A ∪ B = {1, 2, 3, 4, 5, 6}

(3) A = {cow, man, school, town}; B = {school, orange, square}
∴ A ∪ B = {cow, man, school, town, orange, square}

In the following diagram the shaded area represents A ∪ B.

A B



A ∪ B = 

Illustration 8 : If A = {1, 2, 3, 4}; B = {0} and C = ϕ then find
(i) A ∪ B (ii) A ∪ C and (iii) B ∪ C

Solution : Given A = { 1, 2, 3, 4 }; B = {0} and C = ϕ
(i) A ∪ B = { 1, 2, 3, 4, 0 } (ii) A ∪ C = { 1, 2, 3, 4 } (iii) B ∪ C = {0}
for B is a singeton set with "0" as the element.

1.12.2. Intersection of sets : It is the set of all elements which belong to both A and B. The intersection of two sets A and B is denoted by A ∩ B and is read as "A cap B" or "A intersection B"

A ∩ B = {x | x ∈ A and x ∈ B}

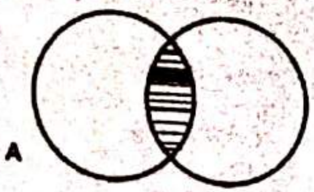
Examples :

(1) A = {1, 2, 3, 4}; B = {3, 4, 5, 6}
A ∩ B = {3, 4}

(2) A = { horse, boy, church, town }
B = {square, church, banana }
A ∩ B = { church }

(3) A = {a, b, c, d }; B = { e, f, g }
A ∩ B = ϕ. In this case, A and B are called disjoint sets.

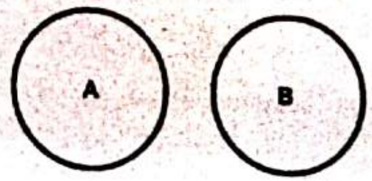
In the following diagram, shaded area represents A ∩ B.



A B

A ∩ B = 

The following diagram shows that A and B are disjoint sets. i.e. they have no common element.



1.12.3. Complement of a set : The complement of a set A is the set of all elements in U which are not in A. Let A be a subset of U. The set of all those elements of U which are not in A is called the complement of A. It is denoted by A′ (read as "A dash").

A′ = U - A
= {x | x ∈ U and x ∉ A}

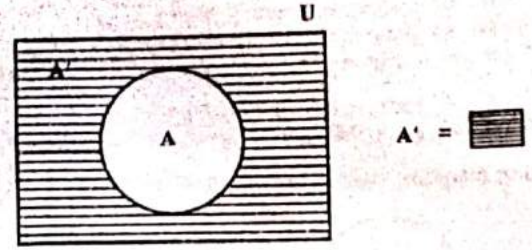
Examples :

(1) U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
A = {0, 2, 4, 6, 8}
∴ A′ = {1, 3, 5, 7, 9}.

(2) U = Set of integers.
A = {Set of negative integers}
A′ = {Set of non-negative integers}

It is clear from the above example that
(i) $A \cup A' = U$ (ii) $A \cap A' = \phi$ (ii) $U' = \phi$ (iv) $\phi' = U$

The following Venn - diagram exhibits the complement of a set A (shaded portion.



1.12.4. Difference of two sets : The difference of two sets A and B is the set of all elements which belong to A but not to B.

It is denoted by A ~ B. Sometimes '~' is also denoted by '-'

$A - B = \{x \mid x \in A \text{ but } x \notin B\}$

Example : $A = \{1, 2, 3, 4\}$; $B = \{2, 3, 5, 6\}$
$A - B = \{1, 4\}$; $B - A = \{5, 6\}$

It is clear from the above example that

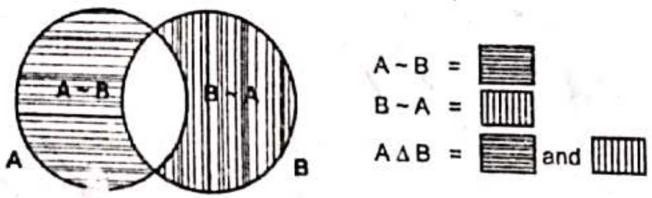
(1) $A \sim B \neq B \sim A$ (2) $(A \sim B) \cap (B \sim A) = \phi$ (3) $A \sim A = \phi$

1.12.5. Symmetric difference : The symmetric difference of two sets A and B is the union of A ~ B and B ~ A. It is denoted by $A \Delta B$.

$A \Delta B = (A \sim B) \cup (B \sim A)$

Example : If $A = \{a, b, c, d, e\}$; $B = \{a, b, e, f, g, h\}$
$A \sim B = \{c, d\}$; $B \sim A = \{f, g, h\}$
$A \Delta B = (A \sim B) \cup (B \sim A)$
$= \{c, d, f, g, h\}$

In the following Venn diagram, the shaded area represents $A \Delta B$.



Note : $A \cup B = (A \Delta B) \cup (A \cap B)$

1.13. Theorem

(a) Union and Intersection of sets are commutative

(i) $A \cup B = B \cup A$ (ii) $B \cap A = A \cap B$

(b) Union and intersection of sets are associative :

If A, B and C are any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

Illustration 9 : If $A = \{-1, 0, 1\}$; $B = \{0, 1, 2\}$; $C = \{2, 3\}$ then prove that (i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

Solution : (i) We have

$A \cup B = \{-1, 0, 1, 2\}$; $B \cup C = \{0, 1, 2, 3\}$

$\therefore A \cup (B \cup C) = \{-1, 0, 1, 2, 3\}$ .... (1)

Also $(A \cup B) \cup C = \{-1, 0, 1, 2, 3\}$ .... (2)

Thus, (1) and (2) imply $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) we have $A \cap B = \{0, 1\}$; $B \cap C = \{2\}$

$A \cap (B \cap C) = \phi$ .... (3)

Also $(A \cap B) \cap C = \phi$ .... (4)

$\therefore$ (3) and (4) imply $A \cap (B \cap C) = (A \cap B) \cap C$.

1.14. Union of sets distributive over intersection of sets :

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Illustration 10 : If $A = \{0, 1, 2\}$; $B = \{2, 3\}$; $C = \{3, 4\}$, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution : We have, $A \cup B = \{0, 1, 2, 3\}$; $A \cup C = \{0, 1, 2, 3, 4\}$ and $B \cap C = \{3\}$

$A \cup (B \cap C) = \{0, 1, 2, 3\}$ .... (1)

Also $(A \cup B) \cap (A \cup C) = \{0, 1, 2, 3\}$ .... (2)

(1) and (2) $\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1.15. Intersection of sets distributive over union of sets:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Illustration 11 : If $A = \{u, v, w\}$; $B = \{w, x\}$ and $C = \{x, y, z\}$, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution : We have $A \cap B = \{w\}$; $A \cap C = \phi$ and $B \cup C = \{w, x, y, z\}$ ....(1)

$\therefore A \cap (B \cup C) = \{w\}$ ....(2)

$(A \cap B) \cup (A \cap C) = \{w\}$ ....

$\therefore$ (1) and (2) $\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

1.16. Reflexive properties : (i) $A \cup A = A$ (ii) $A \cap A = A$

1.17. Complementary properties :

(i) $A \cup A' = U$ (ii) $(A')' = A$ (iii) $(A \cap A') = \phi$

Illustration 12 : Verify the above three results when $U = \{0, 1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$

(i) we have $A' = \{0, 2, 4\}$

$\therefore A \cup A' = \{0, 1, 2, 3, 4, 5\} = U$ (ii) $(A')' = \{1, 3, 5\} = A$ (iii) $(A \cap A') = \phi$

1.18. Identity properties : (i) $A \cup \phi = A$ (ii) $A \cup U = U$ (iii) $A \cap \phi = \phi$ (iv) $A \cap U = A$

1.19. De-Morgan's Law : According to this law (i) the complement of union of two sets is the intersection of their complements.

(ii) The complement of intersection of two sets is the union of their complements.

a) De Morgan's laws for two sets :

(i) $(A \cup B)' = A' \cap B'$ and (ii) $(A \cap B)' = A' \cup B'$

Illustration 13 : If $U = \{0, 1, 2, 3, 4, 5\}$; $A = \{0, 1, 2\}$ and $B = \{2, 4\}$ prove that (i) $(A \cup B)' = A' \cap B'$ and (ii) $(A \cap B)' = A' \cup B'$

Solution : (i) we have $A \cup B = \{0, 1, 2, 4\}$

$\therefore (A \cup B)' = \{3, 5\}$ ....(1)

Also we have $A' = \{3, 4, 5\}$; $B' = \{0, 1, 3, 5\}$

$\therefore A' \cap B' = \{3, 5\}$ ....(2)

From (1) and (2) we get $(A \cup B)' = A' \cap B'$

(ii) we have $A \cap B = \{2\}$

$\therefore (A \cap B)' = \{0, 1, 3, 4, 5\}$ ....(3)

Also we have $A' \cup B' = \{0, 1, 3, 4, 5\}$ ....(4)

From (3) and (4) we get $(A \cap B)' = A' \cup B'$

b) De Morgan's laws for three sets :

(i) $(A - B) \cap (A - C) = A - (B \cup C)$

(ii) $(A - B) \cup (A - C) = A - (B \cap C)$

n(A) = 4

Illustration 14 : If $A = \{1, 2, 3, 4\}$; $B = \{1, 3\}$; $C = \{1, 2, 3\}$ prove that (i) $(A - B) \cap (A - C) = A - (B \cup C)$ and (ii) $(A - B) \cup (A - C) = A - (B \cap C)$

Solution : (i) $A - B = \{2, 4\}$; $A - C = \{4\}$.

$\therefore (A - B) \cap (A - C) = \{4\}$ ....(1)

Also we have $B \cup C = \{1, 2, 3\}$

$A - (B \cup C) = \{4\}$ ....(2)

From (1) and (2) we get $(A - B) \cap (A - C) = A - (B \cup C)$

(ii) we have $(A - B) \cup (A - C) = \{2, 4\}$ ....(3)

Also $B \cap C = \{1, 3\}$

$\therefore A - (B \cap C) = \{2, 4\}$ ....(4)

From (3) and (4) we get $(A - B) \cup (A - C) = A - (B \cap C)$

1.20. Number of elements in a finite set : (Cardinality of a set)

The positive integer, attached to a finite set, to indicate the number of elements of the set is called the cardinality of the set.

It is denoted as n (A) or |A|. For example if $A = \{a, b, c, d\}$ then $n(A) = |A| = 4$.

The cardinary of sets like $A \cup B$, $A \cap B$, $A - B$ etc can be calculated by using the following formula.

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(3) $n[(A \cup B)'] = n(U) - n(A \cup B)$

(4) $n[(A \cup B \cup C)'] = n(U) - n(A \cup B \cup C)$

(5) $n(A - B) = n(A) - n(A \cap B)$

1.21. Remark : If $n(A) = k$, then $n(\rho(A)) = 2^k$.

For example let $A = \{1, 2, 3\}$ $2^n = 2^3 = 2 \times 2 \times 2 = 8$

Here $n(A) = 3$.

and $\rho(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}, \phi\}$

$\therefore n(\rho(A)) = 8 = 2^3$.

As another example, let $B = \{a, b\}$

Here $n(B) = 2$ and $\rho(B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\therefore n(\rho(B)) = 4 = 2^2$

If $C = \phi$, then $\rho(C) = \{\phi\}$.

$\therefore n(C) = 0$ and $n(\rho(C)) = 1 = 2^0$.

Thus if A is a set with k elements, then ρ (A) is a set with 2^k elements.

1.22. Miscellaneous Illustrations :

Illustration 15 : Express the following sets by the Roaster Method.

a) The set G of first four letters of Greek Alphabet.

b) The set F of all positive proper fractions, with denominator 5.

c) $A = \{x^2 \mid x \in N\}$

d) $S = \{x \mid x^2 - 9x + 20 = 0\}$

e) $P = \{x \mid x \in N, x^2 < 50\}$

Solution :

(a) $G = \{\alpha, \beta, \gamma, \delta\}$

(b) $F = \{1/5, 2/5, 3/5, 4/5\}$

(c) $A = \{1, 4, 9, 16, \ldots\}$

(d) $S = \{x \mid x^2 - 9x + 20 = 0\}$

$= \{4, 5\}$, since 4, 5 are roots of the equation $x^2 - 9x + 20 = 0$

(e) $P = \{x \mid x \in N, x^2 \leq 50\}$

$= \{1, 2, 3, \ldots, 7\}$.

Illustration 16 : Write down the following sets in Tabulation Method.

(i) Set of all natural numbers divisible by 7 and are less then 100.

(ii) Set of all prime numbers between 18 and 51.

Solution : (i) The natural numbers divisible by 7 are, 7, 14, 21, ... Since the numbers are less than 100 we have,

$S = \{7, 14, 21, \ldots, 98\}$.

(ii) We know that prime number 'p' is an integer > 1, and is divisible by '1' and 'p' only. Thus, 2, 3, 5, 7, ... are prime numbers. Hence the required set is : $P = \{19, 23, 29, 31, 37, 41, 43, 47\}$

Illustration 17 : Express the following sets in Rule Defining Method.

(a) $M = \{1, 2, 3, 4, 5, 6\}$

(b) $E = \{6, 12, 18, 24, 30, 36, 42, 48, 54\}$

(c) I = Set of all integers between - 15 and 15.

Solution :

(a) we have $M = \{1, 2, 3, 4, 5, 6\}$

$= \{x \mid x$ is a natural number $\leq 6\}$

$= \{x \mid x \in N, x \leq 6\}$.

b) we have $E = \{6, 12, 18, 24, 30, 36, 42, 48, 54\}$

$= \{x \mid x$ is a multiple of 6 and $x \leq 54\}$

$= \{6x \mid x \in N, 1 \leq x \leq 9\}$

(c) we have I = set of all integers between - 15 and +15

$= \{x \mid x \in Z, -15 < x < 15\}$

Illustration 18 : If $A = \{9, 10, 11\}$ and $B = \{x \mid x$ is an integer and $8 < x < 13\}$, which of the following is true ?

(a) $A \subset B$ (b) $B \subset A$ (c) $A = B$

Solution : $A = \{9, 10, 11\}$ $B = \{x \mid x \in Z$ and $8 < x < 13\}$

$= \{9, 10, 11, 12\}$

∴ we conclude that (a) $A \subset B$ is true.

Illustration 19 : Given $A = \{\{a, b\}, \{b, c\}, d, e, \{f\}\}$.

State true or false for the following statements.

(a) $\{a, b\} \subset A$ (b) $\{a, b\} \in A$ (c) $d \in A$ (d) $d \subset A$ (e) $f \in A$
(f) $\{f\} \in A$ (g) $\{f\} \subset A$ (h) $\{d, e\} \subset A$ (i) $a \in A$

Solution :

(a) $\{a, b\} \subset A$ - False, for $\{a, b\}$ is an element of A

(b) $\{a, b\} \in A$ - True (c) $d \in A$ - True

(d) $d \subset A$ - False (e) $f \in A$ - False

(f) $\{f\} \in A$ - True (g) $\{f\} \subset A$ - False

(h) $\{d, e\} \subset A$ - True (i) $a \in A$ - False

Illustration 20 : Write down the subsets of (a) $\{\phi, \{\phi\}\}$ and (b) $\{1, \{2, 3\}\}$

Solution : (a) The subsets of the set $\{\phi, \{\phi\}\}$ are : $\{\phi, \{\phi\}\}, \{\phi\}, \{\{\phi\}\}$ and ϕ.

(b) The subsets of the set $\{1, \{2, 3\}\}$ are : $\{1, \{2, 3\}\}, \{1\}, \{\{2, 3\}\}$ and ϕ.

Illustration 21 : Which of the following sets are singleton and which of them are null ?

(a) $A = \{x \mid x + 5 = 5\}$ (b) $A = \{x \mid (x - 1 = 0$ and $x - 2 = 0\}$

(c) $A = \{0\}$ (d) $A = \{x \mid (x - 1)^2 = 0\}$

Solution : (a) $A = \{x \mid x + 5 = 5\}$ $x + 5 = 5. \therefore x = 5 - 5 = 0$

$= \{0\}$ is a singleton set.

(b) $B = \{x \mid x - 1 = 0 \text{ and } x + 2 = 0\}$

There is no x which satisfies both equations $x - 1 = 0$ and $x + 2 = 0$ simultaneously. Therefore, the set B has no elements. Hence $B = \phi$, a null set.

(c) $A = \{0\}$. Since there is only one element, the set $A = \{0\}$ is a singleton set.

(d) $A = \{x \mid (x - 1)^3 = 0\}$

$(x - 1)^3 = 0 \Rightarrow (x - 1)(x - 1)(x - 1) = 0$

$\Rightarrow x = 1 \text{ or } x = 1 \text{ or } x = 1$

The value 1 is repeated three times.

$\therefore$ The solution of set is $A = \{1\}$, which is a singleton set.

Illustration 22 : State whether the following statements are correct or not ?

(a) If $A = \{2, 3\}$; $B = \{3, 2\}$ and $C = \{x \mid x^2 - 5x + 6 = 0\}$ then $A = B = C$.

Solution : $A = \{2, 3\}$; $B = \{3, 2\}$

$C = \{x \mid x^2 - 5x + 6 = 0\}$

$= \{x \mid (x - 3)(x - 2) = 0\} = \{3, 2\}$

Since the elements of A, B and C are all same, we conclude that $A = B = C$.

(b) If $A = \{x \mid x^2 - 3x + 2 = 0\}$. $B = \{x \mid (x - 1)^2 (x - 2)^3 = 0\}$ then $A \subset B$.

Solution : $A = \{x \mid x^2 - 3x + 2 = 0\}$.

$= \{x \mid (x - 2)(x - 1) = 0\}$

$= \{2, 1\}$

$B = \{x \mid (x - 1)^2 (x - 2)^3 = 0\}$

$= \{x \mid (x - 1)(x - 1)(x - 2)(x - 2)(x - 2) = 0\}$

$= \{x \mid x = 1, 1, 2, 2, 2\}$

$= \{1, 2\}$

Since sets A and B have the same elements, $A = B$, therefore $A \subset B$ is false.

Illustration 23 : Which of the following sets are finite ? Which are infinite ?

(i) The set of integers greater than 100

$A = \{101, 102, 103, 104 \ldots\}$

It is an infinite set.

(ii) Set of human beings in the world. It is a finite set.

(iii) Set of perpendiculars to a given line. It is an infinte set.

(iv) $A = \{x \mid x \text{ is a rational number}\}$

$= \{x \mid x = p/q;\ p, q \in Z,\ q \neq 0\}$

It is an infinite set.

(v) $A = \{x \mid x \text{ is a positive divisor of } 20\}$

$= \{x \mid x = 1, 2, 4, 5, 10, 20\}$

It is a finite set.

Illustration 24 : If $A = \{1, 2, 3\}$; $B = \{1, 2, 3, 4, 5\}$ and $C = \{3, 4, 5, 6\}$ find (i) $A \Delta B$ (ii) $A \cup (B \Delta C)$ (iii) $(A \Delta B) \cap C$

Solution : Given $A = \{1, 2, 3\}$; $B = \{1, 2, 3, 4, 5\}$ and $C = \{3, 4, 5, 6\}$

(i) $A \Delta B = (A - B) \cup (B - A)$

$A - B = \phi$; $B - A = \{4, 5\}$

Therefore $A \Delta B = (A - B) \cup (B - A) = \{4, 5\}$

(ii) $A \cup (B \Delta C)$

$B \Delta C = (B - C) \cup (C - B)$

$= \{1, 2\} \cup \{6\} = \{1, 2, 6\}$

$A \cup (B \Delta C) = \{1, 2, 3\} \cup \{1, 2, 6\} = \{1, 2, 3, 6\}$

(iii) $(A \Delta B) \cap C$

we have $(A \Delta B) = (A - B) \cup (B - A)$

$= \phi \cup \{4, 5\} = \{4, 5\}$

$(A \Delta B) \cap C = \{4, 5\} \cap \{3, 4, 5, 6\} = \{4, 5\}$

Illustration 25 : Given $U = \{a, b, c, d, e, f\}$, $A = \{a, b, c\}$, $B = \{b, c, d, e\}$ and $C = \{e, f\}$. Find $(A \cap B) \cap (A' \cup B) \cap (A \cup B')$

Solution :

$A \cap B = \{a, b, c\} \cap \{b, c, d, e\}$

$= \{b, c\}$

$A' \cup B = \{d, e, f\} \cup \{b, c, d, e\}$
$= \{b, c, d, e, f\}$
$(A \cup B') = \{a, b, c\} \cup \{a, f\}$
$= \{a, b, c, f\}$
$(A \cap B) \cap (A' \cup B) \cap (A \cup B') = \{b, c\} \cap \{b, c, d, e, f\} \cap \{a, b, c, f\} = \{b, c\}$

Illustration 26 : If $A = \{x \mid x \text{ is an integer}, 1 \le x \le 40\}$ and $B = \{x \mid x \text{ is an integer}, 21 \le x \le 100\}$ find $A \cup B$ and $A \cap B$.

Solution : $A = \{x \mid x \text{ is an integer}, 1 \le x \le 40\}$
$= \{1, 2, 3, 4, 5 ......, 40\}$
$B = \{x \mid x \text{ is an interger } 21 \le x \le 100\}$
$= \{21, 22, 23 ...., 100\}$
$\therefore A \cup B = \{1, 2, 3, 4, ..., 40\} \cup \{21, 22, 23, ..., 100\}$
$= \{1, 2, 3, 4, ..., 100\}$
$(A \cap B) = \{1, 2, 3, ... 40\} \cap \{21, 22, 23, ... 100\}$
$= \{21, 22, ..., 40\}$

Illustration 27 : If $U = \{0, 2, 4, 6, 8, 10\}$; $A = \{2, 4\}$; $B = \{4, 8\}$ then verify the following results. $A - B = A \cap B' = B' - A'$

Solution : We have,

$A = \{2, 4\}$; $B = \{4, 8\}$
$A' = \{0, 6, 8, 10\}$; $B' = \{0, 2, 6, 10\}$
$\therefore A - B = \{2\}$ .... (1)
$A \cap B' = \{2\}$ .... (2)
$B' - A' = \{2\}$ .... (3)

From (1), (2) and (3) we get $A - B = A \cap B' = B' - A'$

Illustration 28 : If $V = \{x \mid x + 2 = 0\}$, $R = \{x \mid x^2 + 2x = 0\}$ and $S = \{x \mid x^2 + x - 2 = 0\}$, state whether V, R and S are equal or not?.

Solution : Given $V = \{x \mid x + 2 = 0\}$
$= \{x \mid x = -2\} = \{-2\}$
$R = \{x \mid x^2 + 2x = 0\}$
$= \{x \mid x(x + 2) = 0\} = \{0, -2\}$
$S = \{x \mid x^2 + x - 2 = 0\}$
$= \{x \mid x^2 + 2x - x - 2 = 0\}$
$= \{x \mid (x + 2)(x - 1) = 0\} = \{-2, 1\}$

Hence we conclude that $V \ne R \ne S$.

Illustration 29 : State whether or not the sets A and B are equal in the following.
(a) $A = \{-2, 2\}$, $B = \{x \mid x^2 - 4 = 0\}$
(b) $A = \{0, 1, 2\}$, $B = \{x \mid x^3 - 3x^2 + 2x = 0\}$

Solution : (a) Given $A = \{-2, 2\}$; $B = \{x \mid x^2 - 4 = 0\}$
$= \{x \mid (x + 2)(x - 2) = 0\}$
$= \{-2, 2\}$
$\therefore A = B$

(b) $A = \{0, 1, 2\}$, $B = \{x \mid x^3 - 3x^2 + 2x = 0\}$
$= \{x \mid x(x^2 - 3x + 2) = 0\}$
$= \{x \mid x(x^2 - 2x - x + 2) = 0\}$
$= \{x \mid x(x - 2)(x - 1) = 0\}$
$= \{0, 2, 1\}$
$\therefore A = B$

Illustration 30 : Let $U = \{a, b, c, d, e, f, x, y, z, w\}$; $A = \{a, b, c, d, e\}$; $B = \{b, d, x, y, z\}$. If $A - B$ is defined as $A \cap B'$, verify that $(A - B)' = A' \cup B$.

Solution : $A - B = A \cap B'$

$(A - B) = \{a, b, c, d, e\} - \{b, d, x, y, z\}$
$= \{a, c, e\}$
$\therefore (A - B)' = \{a, b, c, d, e, f, x, y, z, w\} - \{a, c, e\}$
$\{b, d, f, x, y, z, w\}$.
$A' \cup B = \{f, x, y, z, w\} \cup \{b, d, x, y, z\}$
$= \{b, d, f, x, y, z, w\}$
$\therefore (A - B)' = A' \cup B$.

Illustration 31 : If the universal set $U = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $A = \{3, 4, 5, 6\}$, $B = \{3, 7, 9, 5\}$ and $C = \{6, 8, 10, 12, 7\}$ write down the following sets.

(i) A', (ii) B', (iii) C', (iv) $(A')'$, (v) $(B')'$, (vi) $(A \cup B)'$, (vii) $(A \cap B)'$, (viii) $A' \cup C'$ and (ix) $B' \cap C'$.

Solution : Given

(i) $U = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
$A = \{3, 4, 5, 6\}$
$\therefore A' = \{7, 8, 9, 10, 11, 12, 13\}$

(ii) $B = \{3, 7, 9, 5\}$

$\therefore B' = \{4, 6, 8, 10, 11, 12, 13\}$

(iii) $C = \{6, 8, 10, 12, 7\}$

$C' = \{3, 4, 5, 9, 11, 13\}$

(iv) $A' = \{7, 8, 9, 10, 11, 12, 13\}$

$(A')' = \{3, 4, 5, 6\} = A$

(v) $B' = \{4, 6, 8, 10, 11, 12, 13\}$

$\therefore (B')' = \{3, 5, 7, 9\} = B.$

(vi) $(A \cup B) = \{3, 4, 5, 6\} \cup \{3, 7, 9, 5\}$

$= \{3, 4, 5, 6, 7, 9\}$

$\therefore (A \cup B)' = \{8, 10, 11, 12, 13\}$

(vii) $A \cap B = \{3, 4, 5, 6\} \cap \{3, 7, 9, 5\}$

$= \{3, 5\}$

$\therefore (A \cap B)' = \{4, 6, 7, 8, 9, 10, 11, 12, 13\}$

(viii) $A' \cup C' = \{7, 8, 9, 10, 11, 12, 13\} \cup \{3, 4, 5, 9, 11, 13\}$

$= \{3, 4, 5, 7, 8, 9, 10, 11, 12, 13\}$

(ix) $B' \cap C' = \{4, 6, 8, 10, 11, 12, 13\} \cap \{3, 4, 5, 9, 11, 13\}$

$= \{4, 11, 13\}$

Illustration 32 : In a class consisting of 30 students, 10 students take mathematics, 15 take physics and 10 takes neither. Find how many students offer both mathematics and physics ?

Solution :

Number of students in the class	: n (U)	= 30
Students taking mathematics	: n (A)	= 10
Students taking physics	: n (B)	= 15
Students taking neither maths nor physics	: $n[(A \cup B)']$	= 10

We know, $[n(A \cup B)'] = n(U) - n[(A \cup B)']$

Thus $n[(A \cup B)] = n(U) - [n(A \cup B)']$

$= 30 - 10 = 20$

Applying formula,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$20 = 10 + 15 - n(A \cap B)$

$\therefore n(A \cap B) = 10 + 15 - 20 = 5.$

$\therefore$ No. of students taking both maths and physics = 5.

Illustration 33 : In a village with a population of 3000, it is reported that 2200 people read "The Hindu", 1000 read the "Indian express" and 300 read both. Find how many people read neither.

Solution :

Total population in the village	= n (U)	= 3000
People read The Hindu	= n (A)	= 2200
People read Indian express	= n (B)	= 1000
People read both the news papers	= $n(A \cap B)$	= 300
No. of people read neither the news papers	= $n((A \cup B)')$	

We know, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 2200 + 1000 - 300 = 2900.$

$\therefore n((A \cup B)') = n(U) - n(A \cup B)$

$= 3000 - 2900 = 100$

$\therefore$ Thus no. of people reading neither Hindu nor Indian Express = 100.

Illustration 34 : A survey discloses that 20 students are interested in music, 19 are interested in photography and 10 like sports. Further 15 are interested in music and photography, 5 are interested in music and sports, 3 are interested in photography and sports and 2 are interested in all the three. Find out the total number of students surveyed.

Solution :

and ∩

No. of students interested in music	n (M)	= 20
No. of students interested in photography	n (P)	= 19
No. of students interested in sports	n (S)	= 10
No. of students interested in both music and photography	$n(M \cap P)$	= 15
No. of students interested in both music and sports	$n(M \cap S)$	= 5
No. of students interested in photography and sports	$n(P \cap S)$	= 3
No. of students interested in all three :	$n(M \cap P \cap S)$	= 2

We know the formula,

$n(M \cup P \cup S) = n(M) + n(P) + n(S) - n(M \cap P) - n(M \cap S) - n(P \cap S) + n(M \cap P \cap S)$

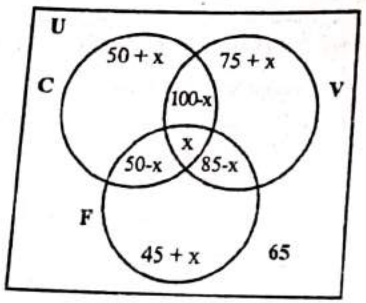
$= 20 + 19 + 10 - 15 - 5 - 3 + 2 = 28.$

Thus the total no. of students surveyed is 28.

Illustration 35 : **In a school of 500 students, 65 do not play any game, 200 play Cricket, 180 play Foot ball and 260 play Volley ball; 50 play both Cricket and Foot ball; 85 play both Foot ball and Volley ball and 100 play Volley ball and Cricket.**

Find : (a) How many students play all the three?
(b) How many play cricket but not others?
(c) How many play foot ball only?
(d) How many play volley ball only? and
(e) How many play only one game? Show by Venn-diagram.

Solution : Let C, F and V represent the sets consisting of students playing cricket, football and volley ball respectively.



Since the number of students playing all the three games is not given, we take this count as x. i.e. n (C ∩ F ∩ V) = x.

Number of students play volley ball and cricket = 100
Area common to V and C but not F has (100 - x) students.
Number of students play foot ball and volley ball = 85
Area common to V and F but not C has (85 - x) students.
Number of students play cricket and foot ball = 50
Area common to C and F but not V has (50 - x) students.
Number of students play volley ball = 260
Number of students play only volley ball

= 260 - (100 - x + x + 85 - x)
= 260 - (185 - x) = 260 - 185 + x
= 75 + x.

Number of students play foot ball = 180
Number of students play only foot ball
= 180 - (50 - x + x + 85 - x)
= 180 - (135 - x) = 180 - 135 + x = 45 + x
Number of students play cricket = 200
Number of students play only cricket
= 200 - (50 - x + x + 100 - x)
= 200 - (150 - x) = 200 - 150 + x = 50 + x

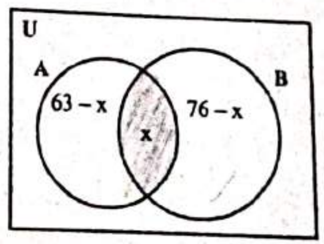
No. of students play atleast one game = Total no. of students - No. of students do not play any of the game.

50 + x + 100 - x + 75 + x + 50 - x + 85 - x + 45 + x + x = 500 - 65 } = 500 - 65
i.e. 405 + x = 435
∴ x = 30

(a) No. of students playing all the three games = 30.
(b) No. of students playing cricket but not others = 50 + x = 50 + 30 = 80.
(c) No. of students playing foot ball only = 45 + x = 45 + 30 = 75.
(d) No. of students playing volley ball only = 75 + x = 75 + 30 = 105.
(e) No. of students play only one game = 80 + 75 + 105 = 260.

Illustration 36 : **In a class of 100 students, 63 like oranges and 76 like apples. How many of them like both ? Show by Venn - diagram.**

Solution : Let A and B represent the sets consisting of students liking orange and apple respectively.



Let the number of students like orange and apple = x.
Number of students like orange = 63.
∴ Number of students like orange alone = 63 - x.
Number of students like apple = 76
∴ Number of students like apple alone = 76 - x.

Here we consider all 100 students like either orange or apple.
Hence $U = A \cup B$

Total number of students $= 63 - x + x + 76 - x$

i.e. $100 = 139 - x \Rightarrow x = 39$.

$\therefore$ Number of students like both orange and apple = 39.

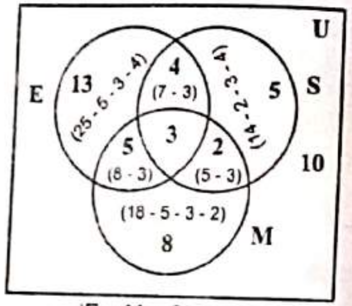
Illustration 37 : In a class of 50 students, the number of students passed in various subjects are as follows.

English 25, Mathematics 18, Science 14, English and Mathematics 8, Mathematics and Science 5, English and Science 7. All the three subjects 3. Using Venn diagram, find the number of students failed in all the subjects.

Solution : Let E, M, S represent the set of students passed in English, Mathematics and Science respectively.

Draw three intersecting circles to repersent E, M and S.

Working Rule :



$(E \cup M \cup S)' = 10$

Allocate the regions from the bottom of the problem.

The no. of students passed in all the 3 subjects = 3.

$\therefore$ Put 3 in the area common to E, M & S.

No. of students passed in English and Science = 7

$\therefore$ No. of students passed in E and S but not M = 7 - 3 = 4.

No. of students passed in Maths and Science = 5

$\therefore$ No. of students passed in M and S and not E = 5 - 3 = 2.

No. of students passed in E & M = 8.

$\therefore$ No. of students passed in E and M but not S = 8 - 3 = 5

No. of students passed in Science = 14

$\therefore$ No. of students passed in Science alone = 14 - (4 + 3 + 2) = 14 - 9 = 5

No. of students passed in Mathematics = 18

$\therefore$ No. of students passed in Maths alone = 18 - (5 + 3 + 2) = 18 - 10 = 8.

No. of students passed in English = 25.

$\therefore$ No. of students passed in English alone (5 + 3 + 4) = 25 - 12 = 13.

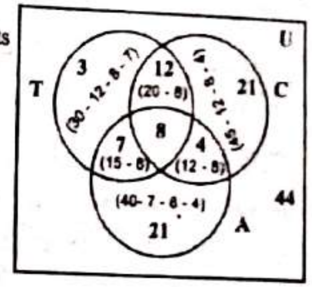
$\therefore$ No. of students passed in any one of the subjects = Sum of all numbers entered in the diagram
= 13 + 4 + 5 + 3 + 5 + 2 + 8 = 40

We know total no. of students = 50.

$\therefore$ No. of students failed in all the subjects = 50 - 40 = 10.

Illustration 38 : In a class consisting of 120 students, 30 take income tax, 40 take accountancy and 45 take costing, 15 students take both income tax and accountancy, 20 take income tax and costing, 12 take accountancy and costing, 8 take all the three subjects (a) How many do not take any of these subjects ? b) How many take only one subject ? c) How many take two subjects ? Show using Venn diagram.

Solution : Let T, A, C represent the sets of students studying Income tax, Accountancy and Costing respectively.



(a) No. of students do not take any of these subjects
$= n(U) - n(T \cup C \cup A)$
$= 120 - (8 + 21 + 4 + 21 + 12 + 7 + 3) = 44$.

(b) No. of students taking only one subject
$= 3 + 21 + 21 = 45$

(c) No. of students taking two subjects only
$= 12 + 4 + 7 = 23$

University Examination Questions & Answers

Illustration 39: If $A = \{1, 2, 3, 4, 5\}$; $B = \{3, 4, 5, 6, 7\}$ and $C = \{1, 2, 6, 7\}$, Find (a) $A \cup (B \cap C)$ (b) $A \cap (B \cup C)$.

[B.Com., M.K.U. April '94]

Solution :

(a) $B \cap C = \{3, 4, 5, 6, 7\} \cap \{1, 2, 6, 7\}$
$= \{6, 7\}$
$\therefore A \cup (B \cap C) = \{1, 2, 3, 4, 5\} \cup \{6, 7\}$
$= \{1, 2, 3, 4, 5, 6, 7\}$

(b) $B \cup C = \{3, 4, 5, 6, 7\} \cup \{1, 2, 6, 7\}$
$= \{1, 2, 3, 4, 5, 6, 7\}$
$\therefore A \cap (B \cup C) = \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 4, 5, 6, 7\}$
$= \{1, 2, 3, 4, 5\}$

Illustration 40: If $A = \{1, 3, 4, 5\}$; $B = \{5, 6, 7, 9\}$ and $C = \{3, 4, 10\}$, Find (a) $(A \cup B) \cap C$ (b) $(A \cap B) \cap C$ (c) $(A \cap B) \cup C$ (d) $(A \cup B) \cup C$.

[B.Com., M.K.U., Nov., '94]

Solution : Given $A = \{1, 3, 4, 5\}$; $B = \{5, 6, 7, 9\}$. $C = \{3, 4, 10\}$

(a) $A \cup B = \{1, 3, 4, 5\} \cup \{5, 6, 7, 9\}$

$= \{1, 3, 4, 5, 6, 7, 9\}$

$\therefore (A \cup B) \cap C = \{1, 3, 4, 5, 6, 7, 9\} \cap \{3, 4, 10\}$

$= \{3, 4\}$

(b) $(A \cap B) = \{5\}$

$\therefore (A \cap B) \cap C = \{5\} \cap \{3, 4, 10\} = \phi$

(c) $(A \cap B) \cup C = \{5\} \cup \{3, 4, 10\}$

$= \{3, 4, 5, 10\}$

(d) $(A \cup B) \cup C = \{1, 3, 4, 5, 6, 7, 9\} \cup \{3, 4, 10\}$

$= \{1, 3, 4, 5, 6, 7, 9, 10\}$

Illustration 41 : If $A = \{1, 3, 4, 5\}$; $B = \{1, 7, 8, 10\}$, find $(A \cup B)$ and $(A \cap B)$

[B.Com., M.K.U. April '95]

Solution : Given $A = \{1, 3, 4, 5\}$; $B = \{1, 7, 8, 10\}$

$A \cup B = \{1, 3, 4, 5\} \cup \{1, 7, 8, 10\}$

$= \{1, 3, 4, 5, 7, 8, 10\}$

$(A \cap B) = \{1, 3, 4, 5\} \cap \{1, 7, 8, 10\}$

$= \{1\}$

Illustration 42: If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then verify that

(a) $A - B = A \cap B' = B' - A'$ and (b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

[B.Com., M.K.U., April '94]

Solution :

(i) $A - B = A \cap B' = B' - A'$

$A - B = \{1, 2, 3, 4\} - \{3, 4, 5, 6\}$

$= \{1, 2\}$ .... (1)

$A \cap B' = \{1, 2, 3, 4\} \cap \{0, 1, 2, 7, 8, 9\}$

$= \{1, 2\}$ .... (2)

$B' - A' = \{0, 1, 2, 7, 8, 9\} - \{0, 5, 6, 7, 8, 9\}$

$= \{1, 2\}$ .... (3)

From, (1), (2) and (3) we conclude that $A - B = A \cap B' = B' - A'$

(ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

$A - B = \{1, 2\}$

$B - A = \{3, 4, 5, 6\} - \{1, 2, 3, 4\} = \{5, 6\}$

$\therefore (A - B) \cup (B - A) = \{1, 2\} \cup \{5, 6\}$

$= \{1, 2, 5, 6\}$ .... (1)

$(A \cup B) = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$

$(A \cap B) = \{3, 4\}$

$\therefore (A \cup B) - (A \cap B) = \{1, 2, 3, 4, 5, 6\} - \{3, 4\}$

$= \{1, 2, 5, 6\}$ .... (2)

Since (1) and (2) are equal, we conclude that

$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Illustration 43 : If $A = \{0, 1, 2\}$, $B = \{1, 3, 4\}$, $C = \{7, 8\}$ are sets find $A \cup (B \cup C)$ and $(A \cup B) \cap C$. Are they equal? **[B.Com., M.K.U. Nov., '94]**

Solution : Given $A = \{0, 1, 2\}$; $B = \{1, 3, 4\}$; $C = \{7, 8\}$

$B \cup C = \{1, 3, 4\} \cup \{7, 8\} = \{1, 3, 4, 7, 8\}$.

$A \cup (B \cup C) = \{0, 1, 2\} \cup \{1, 3, 4, 7, 8\}$

$= \{0, 1, 2, 3, 4, 7, 8\}$ .... (1)

$A \cup B = \{0, 1, 2\} \cup \{1, 3, 4\} = \{0, 1, 2, 3, 4\}$

$(A \cup B) \cap C = \{0, 1, 2, 3, 4\} \cap \{7, 8\}$

$= \phi$ .... (2)

From (1) and (2) we conclude that $A \cup (B \cup C)$ and $(A \cup B) \cap C$ are not equal.

Illustration 44 : If $A = \{1, 3, 4, 5\}$, $B = \{2, 3, 7, 9\}$ are sets and $X = \{1, 2, \ldots, 9\}$ is the universal set, prove that

(a) $(A \cup B)' = (A' \cap B')$ and (b) $(A \cap B)' = A' \cup B'$

[B.Com., M.K.U. Nov., '94]

Solution : Given $X = \{1, 2, \ldots, 9\}$; $A = \{1, 3, 4, 5\}$; $B = \{2, 3, 7, 9\}$

(a) $(A \cup B) = \{1, 3, 4, 5\} \cup \{2, 3, 7, 9\} = \{1, 2, 3, 4, 5, 7, 9\}$

$\therefore (A \cup B)' = \{6, 8\}$ .... (1)

$\therefore A' \cap B' = \{2, 6, 7, 8, 9\} \cap \{1, 4, 5, 6, 8\}$

$= \{6, 8\}$ .... (2)

From (1) and (2) we conclude that $(A \cup B)' = A' \cap B'$

(b) $(A \cap B) = \{1, 3, 4, 5\} \cap \{2, 3, 7, 9\} = \{3\}$

$\therefore (A \cap B)' = \{1, 2, 4, 5, 6, 7, 8, 9\}$ .... (1)

$\therefore A' \cup B' = \{2, 6, 7, 8, 9\} \cup \{1, 4, 5, 6, 8\}$

$= \{1, 2, 4, 5, 6, 7, 8, 9\}$ .... (2)

From (1) and (2) we conclude that $(A \cap B)' = (A' \cup B')$.

Illustration 45 : If $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the universal set and $A = \{3, 4, 5\}$; $B = \{5, 7, 8\}$; $C = \{1, 2\}$ are any sets, find (i) $(A \cup B)'$ (ii) $(A \cap C)'$ and (iii) $(B \cup C)'$.

[B.Com., M.K.U. April '95]

Solution : Given $X = \{1, 2, \ldots\ldots, 8\}$; $A = \{3, 4, 5\}$; $B = \{5, 7, 8\}$, $C = \{1, 2\}$

(i) $(A \cup B) = \{3, 4, 5\} \cup \{5, 7, 8\} = \{3, 4, 5, 7, 8\}$

$\therefore (A \cup B)' = \{1, 2, 6\}$

(ii) $(A \cap C) = \{3, 4, 5\} \cap \{1, 2\} = \phi$

$\therefore (A \cap C)' = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(iii) $(B \cup C) = \{5, 7, 8\} \cup \{1, 2\} = \{1, 2, 5, 7, 8\}$

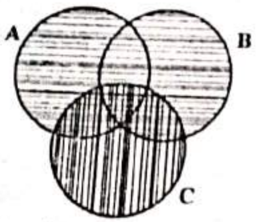
$\therefore (B \cup C)' = \{3, 4, 6\}$.

Illustration 46: Using Venn - diagram, prove that (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

[B.Com., M.K.U. April '92]

Solution : (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

L.H.S.



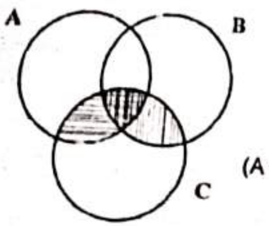
$(A \cup B) \cap C$

$(A \cup B) =$

$C =$

$(A \cup B) \cap C =$ = L.H.S

R.H.S.



$(A \cap C) \cup (B \cap C)$

$(A \cap C) =$

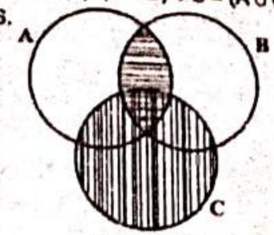
$(B \cap C) =$

$(A \cap C) \cup (B \cap C) =$, & = R.H.S.

L.H.S. = R.H.S. $\therefore (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Solution : (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

L.H.S.



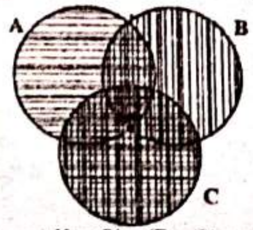
$(A \cap B) \cup C$

$(A \cap B) =$

$C =$

$(A \cap B) \cup C =$ & = L.H.S.

R.H.S.



$(A \cup C) \cap (B \cup C)$

$(A \cup C) =$

$(B \cup C) =$

$(A \cup C) \cap (B \cup C) =$ = R.H.S.

L.H.S. = R.H.S. $\therefore (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Illustration 47 : Draw Venn-diagrams to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[B.Com., M.K.U. April 2003]

Solution : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

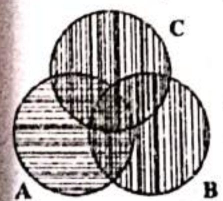


Fig. 1 $A \cap (B \cup C)$

$A =$

$(B \cup C) =$

$A \cap (B \cup C) =$

= L.H.S.

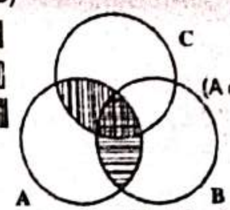


Fig. 2 $(A \cap B) \cup (A \cap C)$

$(A \cap B) =$

$(A \cap C) =$

$(A \cap B) \cup (A \cap C) =$

, & = R.H.S.

L.H.S. = R.H.S. $\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Working Rule :

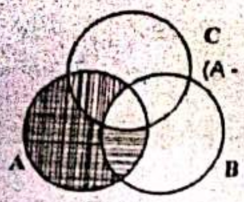
In Fig. 1, A is represented by the horizontally shaded area and $B \cup C$ is represented by the vertically shaded area. $\therefore$ The area shaded with both horizontal and vertical lines represents $A \cap (B \cup C)$

In Fig. 2 the area $(A \cap B)$ is shaded with horizontal lines and $(A \cap C)$ is shaded with vertical lines.

$\therefore$ The area shaded with either vertical or horizontal or both type of lines represent the set $(A \cap B) \cup (A \cap C)$

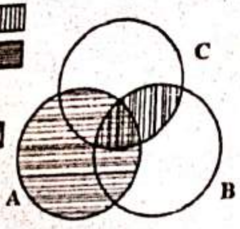
Illustration 48 : Draw a Venn-diagram to verify that (A - B) ∪ (A - C) = A - (B ∩ C)
[B.Com., M.K.U. April 2006]

Solution :



(A - B) =
(A - C) =
(A - B) ∪ (A - C) = , &
= L.H.S.

Fig. 1
(A - B) ∪ (A - C)



A =
(B ∩ C) =
A - (B ∩ C) =
= R.H.S.

Fig. 2
A - (B ∩ C)

L.H.S. = R.H.S. ∴ (A - B) ∪ (A - C) = A - (B ∩ C)

Working Rule :

In Fig. 1, the vertically shaded area represents A - B and the horizontally shaded area represents A - C.

∴ The total shaded area represents (A - B) ∪ (A - C)

In Fig. 2, the horizontally shaded area represents A and the vertically shaded area represents B ∩ C.

∴ The horizontally shaded area represents A - (B ∩ C)

Illustration 49 : In a survey, concerning the reading habits of students, it was found that 60% read magazine A, 50% read magazine B, 50% read magazine C, 30% read A & B, 20% read B & C, 30% read A & C 10% read all the three.

(a) What percentage do not read any of the three magazines?

(b) What percentage read exactly two magazines?

[B.Com., M.K.U. April '94]

Solution : We have,

Percentage of students read magazine	A	:	n (A) = 60
" "	B	:	n (B) = 50
" "	C	:	n (C) = 50
" "	A & B	:	n (A ∩ B) = 30
" "	B & C	:	n (B ∩ C) = 20
" "	A & C	:	n (A ∩ C) = 30
" "	A, B & C	:	n (A ∩ B ∩ C) = 10

(a) **Students do not read any of the three magazines i.e.**

n [(A ∪ B ∪ C)′] = n (U) - n (A ∪ B ∪ C)

n (A ∪ B ∪ C) = n (A) + n (B) + n (C) - n (A ∩ B) - n (B ∩ C) - n (A ∩ C) + n (A ∩ B ∩ C)

= 60 + 50 + 50 - 30 - 20 - 30 + 10 = 90

∴ n [(A ∪ B ∪ C)′] = 100 - 90 = 10

∴ 10% of students do not read any of the three magazines.

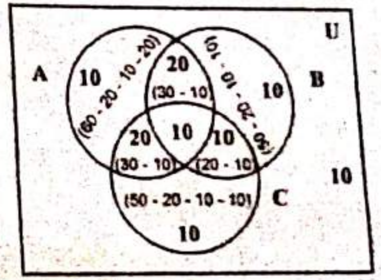
(b) **Percentage of students read only two magazines.**

= n (A ∩ B) - n (A ∩ B ∩ C) + n (B ∩ C) - n (A ∩ B ∩ C) + n (A ∩ C) - n (A ∩ B ∩ C)

= (30 - 10) + (20 - 10) + (30 - 10) = 50

Alternative method :

This problem can also be solved by using Venn diagram.



From the Venn- diagram, we can conclude that

(a) 50% of students read exactly two magazines.

(b) 10% of the students do not read any of the three magazines.

Illustration 50 : If A = {0, 1, 3, 5 }, B = {1, 2, 4, 7} and C = {1, 2, 3, 5, 8 } prove that (A ∪ B) ∪ C = A ∪ (B ∪ C) **[B.Com., M.K.U. Nov. 2005]**

Solution :

(A ∪ B) = {0, 1, 3, 5} ∪ {1, 2, 4, 7 } = {0, 1, 2, 3, 4, 5, 7}

(A ∪ B) ∪ C = {0, 1, 2, 3, 4, 5, 7} ∪ {1, 2, 3, 5, 8}

= {0, 1, 2, 3, 4, 5, 7, 8} .... L.H.S.

(B ∪ C) = {1, 2, 4, 7} ∪ {1, 2, 3, 5, 8} = {1, 2, 3, 4, 5, 7, 8}

A ∪ (B ∪ C) = {0, 1, 3, 5 } ∪ {1, 2, 3, 4, 5, 7, 8}

= {0, 1, 2, 3, 4, 5, 7, 8 } .... R.H.S.

L.H.S. = R.H.S.

∴ (A ∪ B) ∪ C = A ∪ (B ∪ C)

Illustration 51: If A = {1, 2, 3}, B = {2, 3}, C = {3, 4} find $(A \cup B) \cap C$ [B.Com., M.K.U., April 2003]

Solution :

$(A \cup B) = \{1, 2, 3\} \cup \{2, 3\} = \{1, 2, 3\}$

$\therefore (A \cup B) \cap C = \{1, 2, 3\} \cap \{3, 4\} = \{3\}$.

Illustration 52: If A = {2, 3, 6}, B = {3, 8, 10, 11}, C = {6, 8, 10, 12} verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [B.Com., M.K.U., Nov. 1999]

Solution :

$(B \cup C) = \{3, 8, 10, 11\} \cup \{6, 8, 10, 12\} = \{3, 6, 8, 10, 11, 12\}$

$A \cap (B \cup C) = \{2, 3, 6\} \cap \{3, 6, 8, 10, 11, 12\}$

$= \{3, 6\}$ .... L.H.S.

$(A \cap B) = \{2, 3, 6\} \cap \{3, 8, 10, 11\} = \{3\}$.

$(A \cap C) = \{2, 3, 6\} \cap \{6, 8, 10, 12\} = \{6\}$.

$(A \cap B) \cup (A \cap C) = \{3\} \cup \{6\} = \{3, 6\}$ .... R.H.S.

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Illustration 53: If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {1, 5, 6, 7, 8} verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [B.Com., M.K.U., April 2004].

Solution :

$(B \cap C) = \{3, 4, 5, 6\} \cap \{1, 5, 6, 7, 8\} = \{5, 6\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{5, 6\}$

$= \{1, 2, 3, 4, 5, 6\}$ .... L.H.S.

$(A \cup B) = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

$(A \cup C) = \{1, 2, 3, 4\} \cup \{1, 5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 5, 6\} \cap \{1, 2, 3, 4, 5, 6, 7, 8\}$

$= \{1, 2, 3, 4, 5, 6\}$ .... R.H.S.

L.H.S. = R.H.S.

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Illustration 54 : If A = {1, 2, 3} and B = {2, 3, 4, 5} verify (a) $(A \cup B) = (A - B) \cup B$ (b) $A - (A - B) = A \cap B$. [B.Com., M.K.U. Nov. 2003]

Solution :

(a) $(A \cup B) = (A - B) \cup B$

$(A \cup B) = \{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$ .... L.H.S.

$(A - B) = \{1, 2, 3\} - \{2, 3, 4, 5\} = \{1\}$

$(A - B) \cup B = \{1\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$ .... R.H.S.

L.H.S. = R.H.S.

$\therefore (A \cup B) = (A - B) \cup B$

(b) $A - (A - B) = A \cap B$.

L.H.S.

$(A - B) = \{1, 2, 3\} - \{2, 3, 4, 5\} = \{1\}$

$A - (A - B) = \{1, 2, 3\} - \{1\} = \{2, 3\}$ .... L.H.S.

R.H.S.

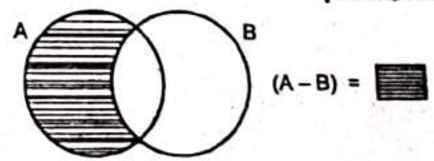
$(A \cap B) = \{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$ .... R.H.S.

L.H.S. = R.H.S.

$\therefore A - (A - B) = A \cap B$

Illustration 55: Draw Venn diagram to represent (a) $(A - B)$ (b) $(A \cap B)'$ [B.Com., M.K.U., April 2003].

Solution : (a) $(A - B)$



(b) $(A \cap B)'$

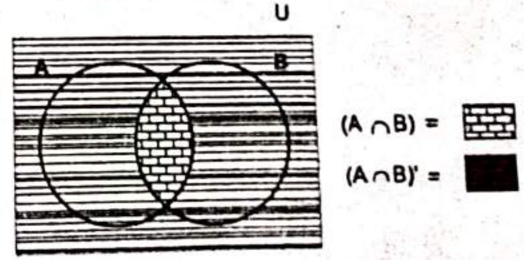
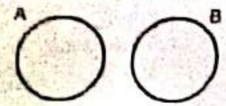
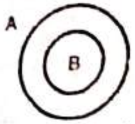


Illustration 56 : Draw Venn diagram for $(A \cap B) = \varphi$ and $(A \cup B) = A$ [B.Com., M.K.U., Nov. 2003]

Solution : (a) $(A \cap B) = \varphi$ (b) $(A \cup B) = A$



$(A \cap B) = \varphi$

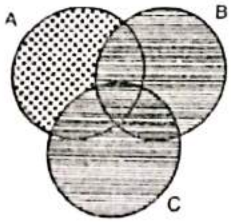


$(A \cup B) = A$

Illustration 57 : Verify using Venn diagram. (a) $A - (B \cup C) = (A - B) \cap (A - C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & (c) $(A \cup B \cup C)' = A' \cap B' \cap C'$ [B.Com., M.K.U., Nov. 2003].

Solution : (a) $A - (B \cup C) = (A - B) \cap (A - C)$

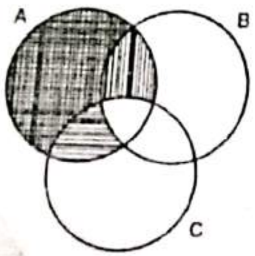
L.H.S.



$(B \cup C)$ =

$A - (B \cup C)$ = ...L.H.S.

R.H.S.



$(A - B)$ =

$(A - C)$ =

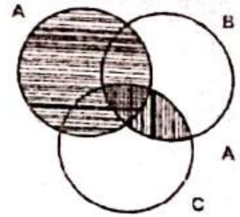
$(A - B) \cap (A - C)$ = ...R.H.S.

=

$\therefore A - (B \cup C) = (A - B) \cap (A - C)$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S.

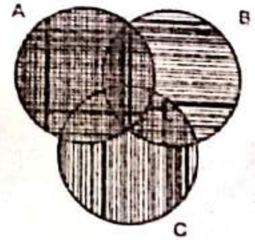


$(B \cap C)$ =

A =

$A \cup (B \cap C)$ = , & .....L.H.S.

R.H.S.



$(A \cup B)$ =

$(A \cup C)$ =

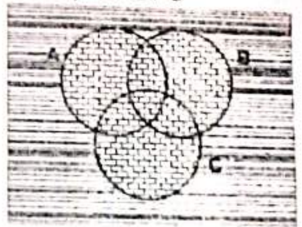
$(A \cup B) \cap (A \cup C)$ =

, & =

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c) $(A \cup B \cup C)' = A' \cap B' \cap C'$

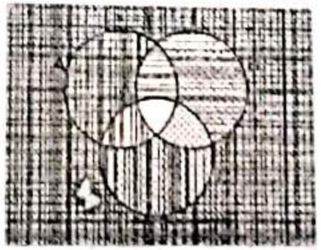
L.H.S.



$(A \cup B \cup C)$ =

$(A \cup B \cup C)'$ = = L.H.S

R.H.S.



A' = B' =

C' = $A' \cap B' \cap C'$ =

= R.H.S.

=

$\therefore (A \cup B \cup C)' = A' \cap B' \cap C'$

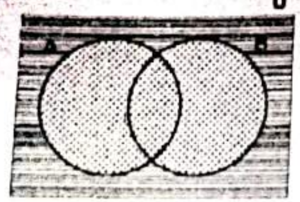
Illustration 58 : State Demorgan's Law by drawing Venn diagram. **[B.Com., M.K.U., April 2003].**

Solution : (a) Demorgan's Law for two sets are :

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(i) $(A \cup B)' = A' \cap B'$

L.H.S.

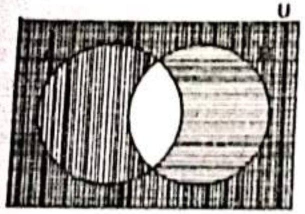


$(A \cup B)'$

$(A \cup B)$ =

$(A \cup B)'$ = ... L.H.S.

R.H.S.



$A' \cap B'$

A' =

B' =

$A' \cap B'$ = ... R.H.S.

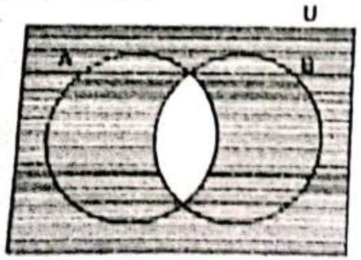
=

L.H.S. = R.H.S.

$\therefore (A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

.H.S.

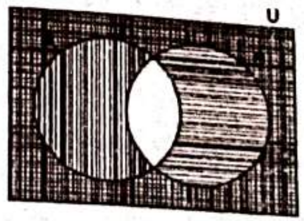


$(A \cap B)'$

$(A \cap B)$ =

$(A \cap B)'$ = ... L.H.S.

R.H.S.



A' =

B' =

$A' \cup B'$ = , & ........R.H.S.

= , &

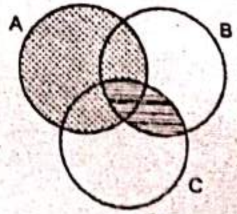
$\therefore (A \cap B)' = A' \cup B'$

b) Demorgan's Law for three sets are (i) $A-(B \cap C) = (A-B) \cup (A-C)$

(ii) $A-(B \cup C) = (A-B) \cap (A-C)$ (Refer Ill. No. 57)

(i) $A-(B \cap C) = (A-B) \cup (A-C)$

L.H.S.

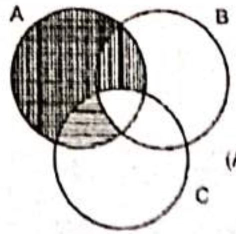


$A-(B \cap C)$

$(B \cap C)$ =

$A-(B \cap C)$ = ...L.H.S.

R.H.S. $(A-B) \cup (A-C)$



$(A-B) \cup (A-C)$

$(A-B)$ =

$(A-C)$ =

$(A-B) \cup (A-C)$ = , & ......R.H.S.

= , &

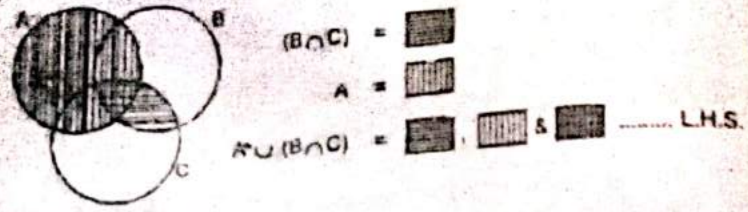
$\therefore A-(B \cap C) = (A-B) \cup (A-C)$

Illustration 59: Using Venn diagram, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

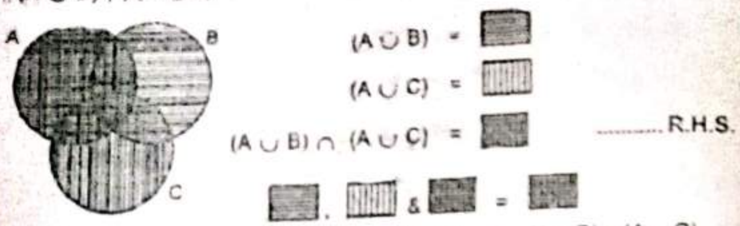
[B.Com., M.K. Univ. April 2004]

Solution: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S



R.H.S. $(A \cup B) \cap (A \cup C)$



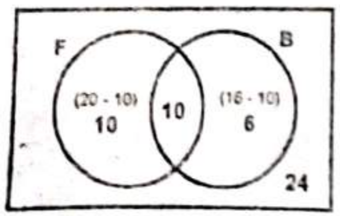
$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Illustration 60: In a class of 50 students, 20 play foot ball, 16 play basket ball and 10 play both. Find following. (a) How many play exactly one? (b) How many play neither? (c) How many do not play foot ball?

[B.Com., M.K. Univ. April 2006]

Solution :

No. of students in a class	:		U = 50
No. of students playing Foot ball	:		F = 20
No. of students playing Basket ball	:		B = 16
No. of Students play both	:		= 10
No. of Students play Football only	:	20 – 10	= 10
No. of Students play Basketball only	:	16 – 10	= 6.



(a) No. of students play exactly one game = 10 + 6 = 16

(b) No. of students play neither Foot ball nor basket ball = 50 – (10+10+6) = 24.

(c) No. of students do not play foot ball = 50 – 20 = 30.

EXERCISE

I State True or False

1. A null set is a subset of every set
2. A set having no element is called singleton set
3. In the set $A = \{1, 2, \{3\}\}$, $\{3\}$ is a sub set of set A
4. For the set $B = \{a, b, c, d\}$, $\{b, c\}$ is a sub set of B
5. The sets $A = \{2, 3, 5, 7\}$, B = {the set of prime numbers ≤ 10}
6. $A \cup B = \{x \mid x \in A \text{ and } x \in B\}$
7. $A \cup A' = U$
8. $A \cap A' = \varphi$
9. $(A \cup B)' = A' \cap B'$
10. A set is a subset of its power set
11. If U is the universal set, then $U' = U$
12. $A - B = B' - A'$
13. $A \cup (B \cup C) = (A \cup B) \cap C$
14. $A \cap (B \cap C) = (A \cap B) \cap C$
15. $A \cup B = B \cap A$

II Choose the correct answer.

1. If A = { set of vowels in English Alphabets}, B = {set of Alphabets in English}, then
   (a) $A \subseteq B$ (b) $A \supseteq B$ (c) $B \subseteq A$ (d) none
2. If $A \subseteq B$ and $B \subseteq C$, then
   (a) $A \supseteq C$, (b) $A \subseteq C$ (c) $A \subset C$ (d) none
3. If there are 3 elements in a set A, then the number of subsets for the set A is
   (a) 3 (b) 9 (c) 27 (d) 6
4. The number of subsets for the set {3,7,1,5} is
   (a) 8 (b) 32 (c) 16 (d) 64
5. If $A = \{2, 4, 5\}$, $B = \{x \mid 3x = 12\}$, which of the following is true.
   (a) $A \subset B$ (b) $B \subset A$ (c) $A = B$ (d) none
6. A void set has
   (a) one element (b) two elements (c) three elements (d) no element