Unit 4:

The countability Axioms

Definition: A space X is said to have a countable basis at x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains atleast one of the elements of $\mathcal{B}$.

A space that has a countable basis at each of its points is said to satisfy the first countability axiom or to be first countable.

Theorem Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first countable

b) Let $f: X \to Y$. If f is continuous then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first countable

The proof is a direct generalization of the proof given in sequence lemma under the hypothesis of metrizability.

<u>Definition</u>: If a space X has a countable basis for its topology, then X is said to have satisfy the second countability axiom or to be second-countable.

<u>Theorem</u> A subspace of a first countable space is first countable and a countable product of first countable space is first countable. A subspace of a second countable spaces is second countable.

<u>proof</u>:-

Consider the second countability axiom. If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the sub-space A of X.

If $\mathcal{B}_i$ is a countable bases for the space X_i then the collection of all products $\prod U_i$ where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$

The proof of the first countability axiom is similar.

<u>Definition</u>: A subset A of a space X is said to be dense in X if $\overline{A} = X$

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Theorem Suppose that X has a countable basis. Then

a) Every open covering of X contains a countable subcollection covering X.

b) There exists a countable subset of X that is dense in X.

Proof: Let $\{B_n\}$ be a countable basis for X.

a) Let $\mathcal{A}$ be an open covering of X. For each positive integer n, choose an element A_n of $\mathcal{A}$ containing the basis element B_n.

Let $\mathcal{A}'$ be a collection of all the sets A_n is countable.

Claim $\mathcal{A}'$ covers X.

Given a point $x \in X$, choose an element A of $\mathcal{A}$ containing x.

Since A is open, there is a basis element B_n such that $x \in B_n \subset A$.

$\therefore x \in B_n \subset A_n$.

$x \subset \cup A_n$.

$\therefore X = \cup A_n$.

$\therefore \mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

b) From each non-empty basis element B_n, choose a point x_n.

Let D be the set consisting of the points x_n

Claim: D is dense in X

Let U be a neighborhood of x.

$\therefore \exists$ B_n such that $x \in B_n \subset U$

$x_n \in B_n \subset U$

$U \cap D \neq \phi$ $(\because x_n \in D)$

$\therefore$ every neighborhood of x intersects D

$\therefore$ D is dense in X

Definition: A space for which every open covering contains countable subcovering is called a Lindelof space.

Definition: A space having a countable dense subset is called separable

example: The space $\mathbb{R}_l$ satisfies all the countability axioms but the second.

soln:

Given $x \in \mathbb{R}_l$

The set of all basis elements of the form $[x, x+\frac{1}{n})$ is a countable basis at x and the rational numbers are dense in $\mathbb{R}_l$.

(ie) $\mathbb{Q}$ is dense in $\mathbb{R}_l$

$\therefore$ $\overline{\mathbb{Q}} = \mathbb{R}_l$.

To prove that $\mathbb{R}_l$ has no countable bases

Let $\mathcal{B}$ be a basis for $\mathbb{R}_l$.

Choose for each x an element $B_x \in \mathcal{B}$ such that

$x \in B_x \subseteq [x, x+\frac{1}{n})$.

If $x \neq y$ then $B_x \neq B_y$.

Since $x = g.l.b$ of B_x & $y = glb\ B_y$

$\therefore B$ is uncountable
$(\because \exists$ a 1-1, $R_l \to B)$

To prove R_l is Lindelof.

(ii) to prove every open covering of R_l by basis elements contains a countable subcollection covering R_l

Let $\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$ be a covering of R by basis elements for lower limit topology.

To prove that $\mathcal{A}$ contains countable sub collection that covers R

Let C be the set $C = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha)$

considering the subspace of R then satisfies 2^{nd} countability axiom because the collection $\{[a_\alpha, b_\alpha]\}$ consists of set open in C. it must contains a countable subcollection that covers C (say) of the elts (a_α, b_α) for $\alpha = \alpha_1 \cdot \alpha_2 - - -$.

then the collection $\mathcal{A}' = \{[a_\alpha, b_\alpha] \mid \alpha = \alpha_1 \cdots\}$ also covers C.

To prove the set $R - C$ is countable

choose for each point of $R - C$ an element of $\mathcal{A}$. containing it at joining these elements of $\mathcal{A}'$ one containing $\mathcal{A}$ countable subcollection of $\mathcal{A}$ that covers all of R_l.

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So let x be a point of $R-C$ 6

$x = a_\alpha$ for some $\alpha \in J$.

Choose q_x to be rational numbers belonging to the interval (a_α, b_α) which is contained in C. So is the interval (x, q_x) it follows that the fcn $x \longrightarrow q_x$ is an injection of $R-C$ into the set Q of rational numbers. So $R-C$ is countable.

<u>example</u>: The product of two lindelof space need not be lindelof.

<u>soln</u>: Let R_l be lindelof.

To prove $R_l \times R_l$ is lindelof.

The basis for $R_l \times R_l$ is $\mathcal{B} = \{([a,b) \times [c,d)\}$

consider the subspace $L = \{x \times (-x) \mid x \in R_l\}$ of $R_l \times R_l$.

<u>Claim</u> $R_l^2 - L$ is open in R_l^2

Let $c \times d \in R_l^2 - L$

$c \times d \notin L$

$d \neq -c$.

then $c \times d \in [c-\epsilon, c+\epsilon] \times [d-\epsilon, d+\epsilon]$

Take $\epsilon = \dfrac{d-(-c)}{4} = \dfrac{d+c}{4}$.

for $x \times y \in [c-\epsilon, c+\epsilon] \times [d-\epsilon, d+\epsilon]$

$\Rightarrow c-\epsilon \leq x \leq c+\epsilon, \; d-\epsilon \leq y \leq d+\epsilon$

$\Rightarrow c-\epsilon+d-\epsilon \leq x+y \leq c+\epsilon+d+\epsilon$

$\Rightarrow c+d-2\epsilon \leq x+y \leq c+d+2\epsilon$

$\Rightarrow 4\epsilon - 2\epsilon \leq x+y \leq 4\epsilon+2\epsilon$

$2\epsilon \leq x+y \leq 6\epsilon.$

$\Rightarrow x+y \neq 0 \Rightarrow y \neq x$

$\therefore$ for each point $c \times d \in R_l^2 - L$

$\exists$ an open ~~cover~~ set $[c-\epsilon, c+\epsilon] \times [d-\epsilon, d+\epsilon]$ containing $c \times d$ $\ni: [c-\epsilon, c+\epsilon] \times [d-\epsilon, d+\epsilon] \subset R_l^2 - L.$

$\therefore R_l^2 - L$ is open in R_l^2 for each point $a \times (-a)$ in L

Consider the open set $[a, x) \times [-a, -y)$ containing the only one point $a \times (-a)$ in L. The collection $\{[a, x) \times [-a, -y)\}$ is an open cover for L.

L is uncountable.

$\therefore$ The open cover $\{[a, x) \times [-a, -y)\}$ is uncountable cover

$L \subset \cup \{[a, x) \times [-a, -y]\}$

$\therefore R_l \times R_l = L \cup (R_l^2 - L)$

$\Rightarrow \cup \{[a, x] \times [-a, -y)\} \cup R_l^2 - L$

W.K.T $R_l^2 - L$ is open in R_l^2

$\therefore$ The collection $\{[a, x) \times [-a, -y)\}$ $\cup (R_l^2 - L)$ is an open cover for R_l^2.

This has no countable subcollection

covering $\mathbb{R}_l^2$

Since for each pt $a \times (-a)$ in L

there is only one open set $[a, x] \times [-a, -y]$ containing $a \times (-a)$

$\therefore$ $\mathbb{R}_l^2$ is not lindelof.

The Separation Axioms

Definition : Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open set containing x and B respectively.

The space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exists disjoint open sets containing A and B respectively.

Lemma: Let X be a topological space. Let one-point sets in X be closed, (a) X is regular if and only if given a point x of X and a neighborhood U of x there is a nbh V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is open set V containing such that $\overline{V} \subset U$.

a) Suppose X is regular. Let $x \in X$ and the neighborhood U of x are given.

Let $B = X - U$.

Now B is closed and $x \notin B$

Since X is regular, there are disjoint open sets V and W such that $x \in V$ and $B \subset W$.

$\therefore V \cap W = \phi$.

$\Rightarrow V \subset W^c$

$\Rightarrow \overline{V} \subset W^c$ ($\because W^c$ is closed)

closure - intersection of all closed sets con-

Also $B \subset W$.

$\therefore W^c \subset B^c = \cancel{U}$ $\Rightarrow W^c \subset U$

$\Rightarrow \overline{V} \subset W^c \subset U$

$\therefore$ ~~Now~~ $\overline{V} \subset U$

Conversely, that the condition holds.

<u>To prove</u> X is regular

Let $x \in X$ and B is a closed set in X such that $x \notin B$.

Let $U = X - B$.

Now U is open in X and $x \in U$

By hypothesis, there is a neighborhood V of x such that $x \in V$ and $\overline{V} \subset U$.

Now $\overline{V} \subset U$.

$\Rightarrow X - U \subset X - \overline{V}$

$\Rightarrow B \subset X - \overline{V}$

Now $X - \overline{V}$ is open in X

$\therefore$ we have ~~two~~ disjoint open set V

and $X-\overline{V}$ containing x and B respectively

$\therefore X$ is regular.

b). Suppose X is normal.
Let A be any closed set in X and U be any open set containing A.
Let $B = X-U$.
Now B is closed and $A \cap B = \phi$.
Since X is normal, there are disjoint open sets V and W. Such that $A \subset V$, $V \subset W$.

$\therefore V \cap W = \phi$
$\Rightarrow V \subset W^c$
$\Rightarrow \overline{V} \subset W^c$

Also $B \subset W$
$\therefore W^c \subset B^c = U$
$\therefore \overline{V} \subset U$

$\therefore$ the condition holds.

Conversely suppose that the condition holds.

To prove X is normal

Let A and B be closed set in X such that $A \cap B =$ [illegible]
Let A and B be closed set in X and $A \subset$ [illegible]
Let $U = X-B$. Now U is open in X and
By hypothesis $\exists$ an open set ~~V~~ V such th[illegible]
$A \subset V$ & $\overline{V} \subset U$

Now $\overline{V} \subset U$
$\Rightarrow$ ~~$X-\overline{V}$~~ $\Rightarrow X-U \subset X-\overline{V}$
$\Rightarrow B \subset X-\overline{V}$

Also $X-\overline{V}$ is open in X.

$\therefore$ we have two disjoint open sets V and $X-\overline{V}$ containing A and B respectively.

$\therefore X$ is normal.

Theorem a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof:

Let X be a Hausdorff space and Y be a subspace of X.

Let ~~$x, y \in X$~~ $x, y \in Y$ such that $y \neq x$

$\Rightarrow x, y \in X$

$\Rightarrow \exists$ disjoint nbh U and V in X of x and y respectively

Then $U \cap Y$ and $V \cap Y$ are disjoint nbh of x and y in Y.

$\therefore Y$ is Hausdorff.

Let $\{X_\alpha\}$ be a family of Hausdorff spaces.

Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$.

Since $x \neq y$ there is some index β such that $x_\beta \neq y_\beta$ in X_β.

Since X_β is Hausdorff $\exists$ disjoint open sets U and V in X_β containing x_β & y_β respectively

Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in πX_α containing x and y.

$\therefore$ πX_α is a Hausdorff space.

b) Let X be a regular space, Y is a subspace of a regular space X.

P.T Y is regular.

Let x be a point of Y and let B be a closed subsets of Y disjoint from x

Now $\bar{B} \cap Y = B$.

where $\bar{B}$ denotes the closure of B in X

[$\because$ $y \in \bar{B} \cap Y \Rightarrow y \in \bar{B} \Rightarrow \forall$ nbh of y intersects $B \Rightarrow y \in \bar{B}$ in $Y = B$

$\therefore$ B is closed.

$\bar{B} \cap Y \subset B$.

Also $B \subset \bar{B} \cap Y \Rightarrow \bar{B} \cap Y = B$]

$\therefore$ $x \notin \bar{B}$

($\because$ $x \in Y$ & $x \notin B \Rightarrow x \notin \bar{B}$)

$\therefore$ $\bar{B}$ is closed is not containing x

$\exists$ disjoint open sets U and V of X containing x ($\because$ X is regular)

Then $U \cap Y$ & $V \cap Y$ are disjoint open set in Y containing x and B respectively

$\therefore$ Y is regular.

Let $\{X_\alpha\}$ be a family of regular space

Let $X = \prod X_\alpha$.

$\Rightarrow X$ is Hausdorff ($\because$ regular $\Rightarrow$ Hausdorff
each x_α regular $\Rightarrow$
each x_α Hausdorff)

$\Rightarrow$ one point sets are closed in X.
($\because \prod X_\alpha$ is Hausdorff)

Let $x = (x_\alpha)$ be a point of X.

Let U be a nbh of x in X

$x \in U$, $\exists$ a basis elt $\prod U_\alpha$ about x
contained in U.
Since X_α is regular and U_α is a nbh containing x_α
choose, for each α a nbh V_α of x_α in X_α
such that $\overline{V_\alpha} \subset U_\alpha$ if it happens $U_\alpha = X_\alpha$.

Chose $V_\alpha = X_\alpha$

Then $V = \prod V_\alpha$ is a nbh of x in X

Since $\overline{V} = \prod \overline{V_\alpha}$

~~But~~

But $\prod \overline{V_\alpha} \subset \prod U_\alpha$

$\overline{V} \subset \prod U_\alpha \subset U$.

$\therefore x \in X$, U is a nbh of x,

$\exists$ a nbh U of x, $\exists$ a nbh V of x such that

$\overline{V} \subset U$ (by above lemma)

$\therefore X$ is regular.

Normal spaces:

Theorem Every regular space with a countable basis is normal.

Proof: Let X be a regular space with a countable basis $\mathcal{B}$.

To prove that X is normal.

Let A and B be disjoint closed subsets of X. (ie) $A \cap B = \phi$

Each point $x \in A$ has a nbh U not intersecting B ($x \notin B$

$x \in X-B$, $x \in U$ $x \notin B$ $U \cap B = \phi$ $X-B$ is open

Since X is regular, there is a nbh V of x whose closure lies in U (ie) $\overline{V} \subset U$.

choose an elt $\mathcal{B}$ containing x and contained in V

(for each $x \in \mathcal{B}$ $\exists$ $B \in \mathcal{B}$ $\ni$: $x \in B \subset U$)

(by defn. here were $x \in V$ $\exists$ $U \in \mathcal{B}$ $\ni$: $x \in U \subset V$)

By choosing such a basis elt for each x in A, we

$\overline{U} \subset \overline{V} \subset U$

$\overline{U} \cap V \subset U \cap V = \phi$

Construct a countable covering of A by open sets

$\overline{U} \cap V = \phi$

whose closures do not intersect B.

Since this covering of A is countable, we can index it with the positive integers, let us denote it by $\{U_n\}$. (covers A)

(covers B)

IIIly choose a countable collection $\{V_n\}$ of open sets covering B such that each set $\overline{V_n}$ is disjoint from A.

$\overline{V_n} \cap A = \phi$

The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B respectively but they need not be disjoint. $A \subset U$ $B \subset V$

Construct two open sets are disjoint. Given n, define

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$$

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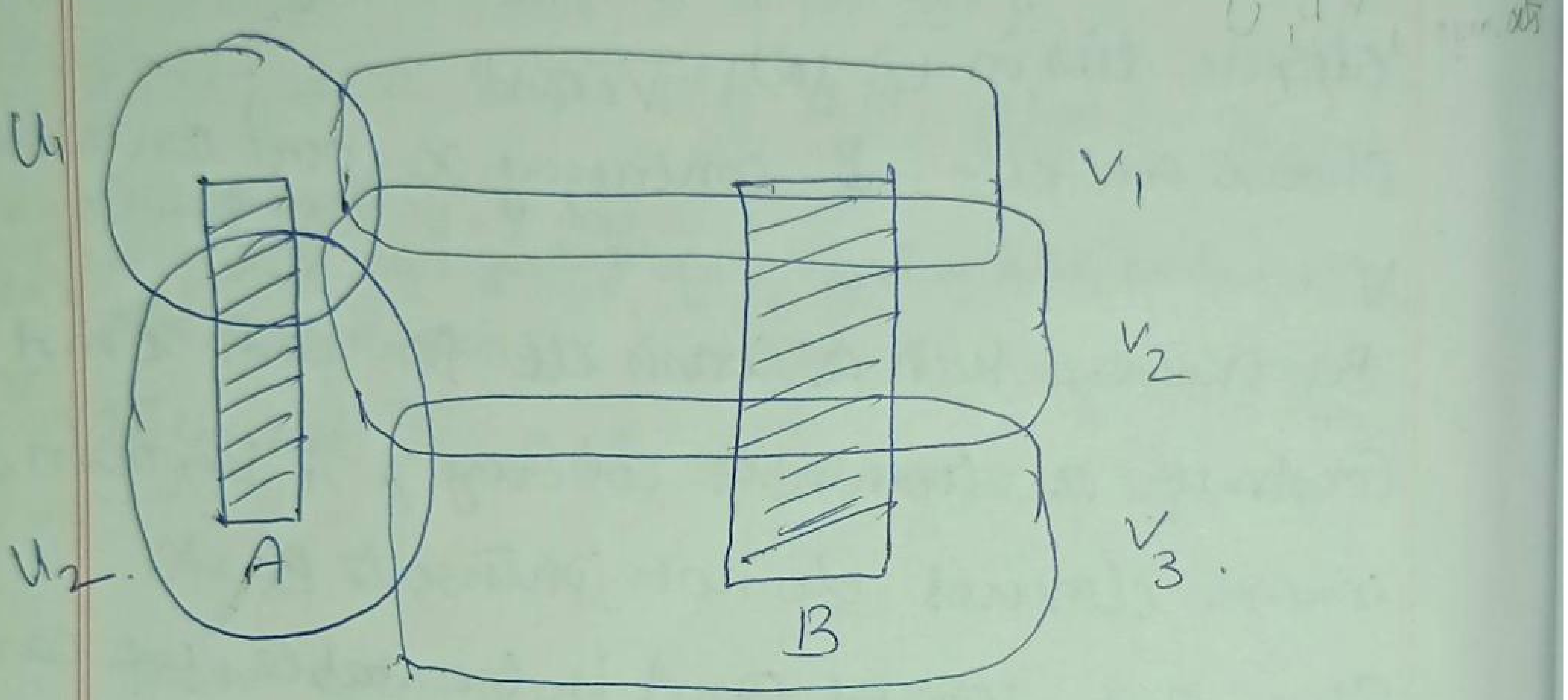
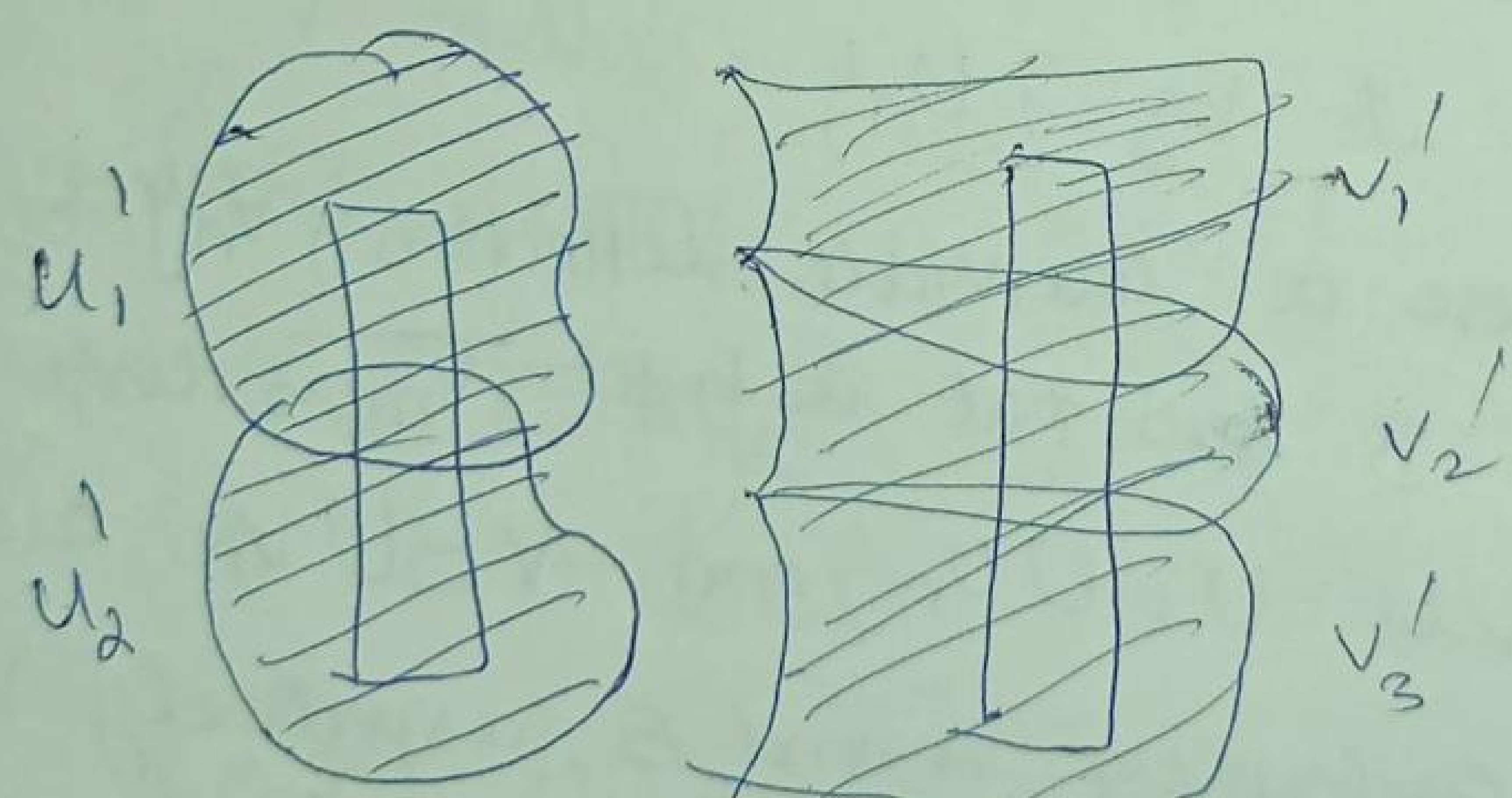
Note that each set U_n' is open, being the difference of an open set U_n and a closed set $\bigcup_{i=1}^{n} \overline{V_i}$

Ill^{ly} each set V_n' is open.

The collection $\{U_n'\}$ covers A because each $x \in A \in U_n$ for some n and x belongs to none of the sets $\overline{V_i}$

Ill^{ly} the collection $\{V_n'\}$ covers B.





Finally the open sets

$$U' = \bigcup_{n \in \mathbb{Z}_+} U_n' \quad V' = \bigcup_{n \in \mathbb{Z}_+} V_n'$$

Since U_n' & V_n' are open then union are open $\therefore U'$ & V' are open

are disjoint.

C.T. U' & V' are disjoint opensets containing A & B
$A \subset U'$ & $B \subset V'$

(ii) P.T $U' \cap V' = \phi$

suppose $U' \cap V' \neq \phi$

$x \in U' \cap V'$.

$\therefore x \in U'$ and $x \in V'$.

$x \in U' \Rightarrow x \in U_j'$ for some j

$x \in V' \Rightarrow x \in V_k'$ for some k

$x \in U_j'$ for some j and $x \in V_k'$ for some k

$\therefore x \in U_j' \cap V_k'$ for some j and k.

case (i) if $j \leq k$

It follows from the definition of U_j' that $x \in U_j$ and since $j \leq k$ it follows that from the definition of V_k' that $x \notin \overline{U_j}$.

A similar contradiction arises if $j \geq k$.

$\therefore U' \cap V' = \phi$

$\therefore X$ is normal

X——X

L.T $A \subset U'$

Let $x \in A$ then $x \in U_n$ for some n.

$\Rightarrow x \notin \overline{V_i}$

$x \in U_n - \bigcup_{i=1}^{n} \overline{V_i}$

$x \in U_n'$

$x \in U'$

$\therefore A \subset U'$

(||)ly $B \subset V'$

Theorem: Every metrizable space is normal

Proof: Let X be a metrizable space with metric d.

To prove that X is normal

Since X is metrizable, X is Hausdorff.

$\therefore$ one point sets are closed in X

Let A and B be disjoint closed subsets of X.

Let $a \in A$ and so $a \in X-B$.

Since B is closed, $X-B$ is open.

$\therefore$ there exists a real number, $\epsilon_a > 0$ such that $B(a, \epsilon_a) \subset X-B$.

$\therefore B(a, \epsilon_a)$ does not intersect B.

$|||^{ly}$ for each $b \in B$, $\exists$ real number $\epsilon_b > 0$ such that $B(b, \epsilon_b)$ does not intersect A.

Define $U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2})$

$$V = \bigcup_{b \in B} B(b, \frac{\epsilon_b}{2})$$

Then U and V are open sets containing A and B.

~~claim~~ Assert that U and V are disjoint

(i.e) $U \cap V = \phi$

Suppose $U \cap V \neq \phi$

Let $z \in U \cap V$.

$z \in U$ and $z \in V$

$z \in U \Rightarrow z \in B(a, \frac{\epsilon_a}{2})$ for some $a \in A$.

$$\therefore d(a,z) < \frac{\epsilon_a}{2}$$

$z \in V \Rightarrow z \in B(b, \frac{\epsilon_b}{2})$ for some $b \in B$

$$\therefore d(b,z) < \frac{\epsilon_b}{2}.$$

Case (i) $\epsilon_a \leq \epsilon_b$.

$$\therefore \frac{\epsilon_a}{2} \leq \frac{\epsilon_b}{2}.$$

$$\begin{aligned} d(a,b) &\leq d(a,z) + d(b,z) \\ &\leq \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2}. \\ &\leq \frac{\epsilon_b}{2} + \frac{\epsilon_b}{2} = \epsilon_b. \end{aligned}$$

$$\therefore d(a,b) \leq \epsilon_b.$$

$$\therefore a \in B(b, \epsilon_b)$$

which is a contradiction $(\because A \cap B(b, \epsilon_b) = \phi)$

Case (ii). $\epsilon_b \leq \epsilon_a$.

$$\frac{\epsilon_b}{2} \leq \frac{\epsilon_a}{2}$$

$$\begin{aligned} d(a,b) &\leq d(a,z) + d(b,z) \\ &\leq \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} = \frac{\epsilon_a}{2} + \frac{\epsilon_a}{2} = \epsilon_a \end{aligned}$$

$$\therefore d(a,b) < \epsilon_a$$

$$b \in B(a, \epsilon_a)$$

which is a contradiction $(\because B \cap B(a, \epsilon_a) = \phi$

$$\therefore U \cap V = \phi$$

$\therefore X$ is normal.

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Theorem: Every compact Hausdorff space is normal

Proof:- Let X be a compact Hausdorff space

T.P.T ~~X~~ is normal.

Since X is hausdorff, one point sets are closed in X

Let A and B be two disjoint closed sets in X

Since X is compact, A and B are closed in X.

$\therefore$ A and B are compact.

Let $a \in A$ $\therefore$ $a \notin B$.

Let $y \in B$, $a \neq y$.

Since X is hausdorff there are disjoint open sets U_y and V_y such that $a \in U_y$ and $y \in V_y$

$$\therefore B \subset \bigcup_{y \in B} V_y$$

$\therefore \{V_y\}_{y \in B}$ is an open set covering of B

Since B is compact.

$B \subset V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$ for some

$y_1, y_2 \cdots y_n \in B$

Let $U_a = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$.

$$V_a = V_{y_1} \cap V_{y_2} \cap \cdots \cap V_{y_n}$$

Now U_a and V_a are open set in X

such that $a \in U_a$ and $B \in V_a$

To prove that $U_a \cap V_a = \phi$

Suppose $U_a \cap V_a \neq \phi$

Let $z \in U_a \cap V_a \Rightarrow z \in U_a$ & $z \in V_a$

$z \in U_a \Rightarrow z \in U_{y_i} \quad \forall \, i = 1 \cdots n.$

$z \in V_a \Rightarrow z \in V_{y_j}$ for some j

$\therefore U_{y_i} \cap V_{y_i} \neq \phi$

$\Rightarrow \Leftarrow \; (\because U_{y_i} \cap V_{y_i} = \phi)$

$\therefore U_a \cap V_a = \phi.$

Thus for each $a \in A$ we obtain a pair of disjoint open sets U_a & V_a such that $a \in U_a$ and $B \in V_a$.

Now $\{U_a\}_{a \in A}$ is an open covering for A

Since A is compact

$A \subset U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$ for some $a_1 \cdots a_n \in A$

Let $U = U_{a_1} \cup \cdots \cup U_{a_n}$

$V = V_{a_1} \cup \cdots \cup V_{a_n}$

Now U and V are open set in $X \ni: A \subset U$ and $B \subset V$.

<u>claim</u> $U \cap V = \phi$.

Suppose $U \cap V \neq \phi$

$z \in U \cap V$.

$z \in U$ & $z \in V$.

$z \in U \Rightarrow z \in U_{a_j}$ for some j

$z \in V \Rightarrow z \in V_{a_i} \quad \forall \, i = 1 \cdots n.$

$\therefore U_{a_j} \cap V_{a_i} \neq \phi$

$\Rightarrow \Leftarrow$

$\therefore U \cap V = \phi$ Hence X is normal